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Constant-Cutoff Approach to the Isovector Exchange Magnetic Moment

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Abstract. We suggest a quantum stabilization method for the SU(2) σ-model based on the constant-cutoff limit of the cutoff quantization method developed by Balakrishna et al., which avoids the difficulties with the usual soliton boundary conditions pointed out by Iwasaki and Ohyama. We investigate the baryon number $B = 1$ sector of the model and show that after the collective coordinate quantization it admits a stable soliton solution which depends on a single dimensional arbitrary constant. We then derive the results for isovector exchange magnetic moment operators for two-nucleon systems in the constant-cutoff approach to the SU(2) σ-model using the product Ansatz for the soliton field operator.

1 Introduction

It was shown by Skyrme [1] that baryons can be treated as solitons of a nonlinear chiral theory. The original Lagrangian of the chiral SU(2) σ-model is

$$\mathcal{L} = \frac{F_\pi^2}{16} \mathrm{Tr}\, \partial_\mu U \partial^\mu U^+ \ , \tag{1.1}$$

where

$$U = \frac{2}{F_\pi}(\sigma + i\boldsymbol{\tau} \cdot \boldsymbol{\pi}) \tag{1.2}$$

is a unitary operator ($UU^+ = 1$) and F_π is the pion-decay constant. In (1.2) $\sigma = \sigma(\boldsymbol{r})$ is a scalar meson field and $\pi = \pi(\boldsymbol{r})$ is the pion-isotriplet.

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The classical stability of the soliton solution to the chiral σ-model Lagrangian requires the additional ad-hoc term, proposed by Skyrme [1], to be added to (1.1)

$$\mathcal{L}_{\text{Sk}} = \frac{1}{32e^2}\text{Tr}\left[U^+\partial_\mu U, U^+\partial_\nu U\right]^2 \tag{1.3}$$

with a dimensionless parameter e and where $[A, B] = AB - BA$. It was shown by several authors [2] that, after the collective quantization using the spherically symmetric Ansatz

$$U_0(\boldsymbol{r}) = \exp\left[i\boldsymbol{\tau}\cdot\hat{\boldsymbol{r}}F(r)\right] \ , \quad \hat{\boldsymbol{r}} = \boldsymbol{r}/r \ , \tag{1.4}$$

the chiral model, with both (1.1) and (1.3) included, gives a good agreement with the experiment for several important physical quantities. Thus it should be possible to derive the effective chiral lagrangian, obtained as a sum of (1.1) and (1.3), from a more fundamental theory like QCD. On the other hand it is not easy to generate a term like (1.3) and give a clear physical meaning to the dimensionless constant e in (1.3) using QCD.

Mignaco and Wulck (MW) [3] indicated therefore a possibility to build a stable single baryon ($n = 1$) quantum state in the simple chiral theory, with the Skyrme stabilizing term (1.3) omitted. MW have shown that the chiral angle $F(r)$ is in fact a function of a dimensionless variable $s = \frac{1}{2}\chi''(0)r$, where $\chi''(0)$ is an arbitrary dimensional parameter intimately connected to the usual stability argument against the soliton solution for the non-linear σ-model Lagrangian.

Using the adiabatically rotated Ansatz $U(\boldsymbol{r},t) = A(t)U_0(\boldsymbol{r})A^+(t)$, where $U_0(\boldsymbol{r})$ is given by (1.4), MW obtained the total energy of the nonlinear σ-model soliton in the form

$$E = \frac{\pi}{4}F_\pi^2\frac{1}{\chi''(0)}a + \frac{1}{2}\frac{[\chi''(0)]^3}{\frac{\pi}{4}F_\pi^2 b}J(J+1), \tag{1.5}$$

where

$$a = \int_0^\infty ds\left[\frac{1}{4}s^2\left(\frac{d\mathcal{F}}{ds}\right)^2 + 8\sin^2\left(\frac{1}{4}\mathcal{F}\right)\right] , \tag{1.6}$$

$$b = \int_0^\infty ds\,\frac{64}{3}s^2\sin^2\left(\frac{1}{4}\mathcal{F}\right) , \tag{1.7}$$

and $\mathcal{F}(s)$ is defined by

$$F(s) = -n\pi + \frac{1}{4}\mathcal{F}(s) \ . \tag{1.8}$$

The stable minimum of the function (1.5), with respect to the arbitrary dimensional scale parameter $\chi''(0)$, is

$$E = \frac{4}{3}F_\pi\left[\frac{3}{2}\left(\frac{\pi}{4}\right)^2\frac{a^3}{b}J(J+1)\right]^{1/4} . \tag{1.9}$$

Despite the non-existence of the stable classical soliton solution to the nonlinear σ-model, it is possible, after the collective coordinate quantization, to build a stable chiral soliton at the quantum level, provided that there is a solution $F = F(r)$ which satisfies the soliton boundary conditions, i.e. $F(0) = -n\pi$, $F(\infty) = 0$ such that the integrals (1.6) and (1.7) exist.

However, as pointed out by Iwasaki and Ohyama [4], the quantum stabilization method in the form proposed by MW [3] is not correct since in the simple σ-model the conditions $F(0) = -n\pi$ and $F(\infty) = 0$ cannot be satisfied simultaneously. In other words if the condition $F(0) = -\pi$ is satisfied Iwasaki and Ohyama obtained numerically $F(\infty) \to -\pi/2$, and the chiral phase $F = F(r)$ with correct boundary conditions does not exist.

Iwasaki and Ohyama also proved analytically that both boundary conditions $F(0) = -n\pi$ and $F(\infty) = 0$ can not be satisfied simultaneously. Introducing a new variable $y = 1/r$ into the differential equation for the chiral angle $F = F(r)$ we obtain

$$\frac{d^2F}{dy} = \frac{1}{y^2}\sin(2F) \tag{1.10}$$

There are two kinds of asymptotic solutions to the equation (1.10) arround the point $y = 0$, which is called a regular singular point if $\sin 2F \approx 2F$. These solutions are

$$F(y) = \frac{m\pi}{2} + cy^2 \ , \qquad m = \text{even integer} \ , \tag{1.11}$$

$$F(y) = \frac{m\pi}{2} + \sqrt{cy}\cos\left[\frac{\sqrt{7}}{2}\ln(cy) + \alpha\right] \ , \qquad m = \text{odd integer} \ , \tag{1.12}$$

where c is an arbitrary constant and α is a constant to be chosen adequately. When $F(0) = -n\pi$ then we want to know which of these two solutions are approached by $F(y)$ when $y \to 0$ ($r \to \infty$)? In order to answer to that question we multiply (1.10) by $y^2F'(y)$, integrate with respect to y from y to ∞ and use $F(0) = -n\pi$. Thus we get

$$y^2F'(y) + \int_y^\infty dy\, 2y[F'(y)]^2 = 1 - \cos[2F(y)] \ . \tag{1.13}$$

Since the left-hand side of (1.13) is always positive, the value of $F(y)$ is always limited to the interval $n\pi - \pi < F(y) < n\pi + \pi$. Taking the limit $y \to 0$, (1.13) is reduced to

$$\int_0^\infty dy\, 2y[F'(y)]^2 = 1 - (-1)^m \ , \tag{1.14}$$

where we used (1.11-12). Since the left-hand side of (1.14) is strictly positive, we must choose an odd integer m. Thus the solution satisfying $F(0) = -n\pi$ approaches (1.12) and we have $F(\infty) \neq 0$. The behaviour of the solution (1.11) in the asymptotic region $y \to \infty$ ($r \to 0$) is investigated by multiplying (1.10) by $F'(y)$, integrating from 0 to y and using (1.11). The result is

$$[F'(y)]^2 = \frac{2\sin^2 F(y)}{y^2} + \int_0^y dy\, \frac{2\sin^2 F(y)}{y^3} \ . \tag{1.15}$$

From (1.15) we see that $F'(y) \longrightarrow$ constant as $y \longrightarrow \infty$, which means that $F(r) \simeq 1/r$ for $r \longrightarrow 0$. This solution has a singularity at the origin and can not satisfy the usual boundary condition $F(0) - -n\pi$.

In [5] the present author suggested a method to resolve this difficulty by introducing a radial modification phase $\varphi = \varphi(r)$ in the Ansatz (1.4), as follows

$$U(\boldsymbol{r}) = \exp\left[i\boldsymbol{\tau} \cdot \boldsymbol{r}_0 F(r) + i\varphi(r)\right] \ . \tag{1.16}$$

Such a method provides a stable chiral quantum soliton but the resulting model is an entirely non-covariant chiral model, different from the original chiral σ-model.

In the present paper we use the constant-cutoff limit of the cutoff quantization method developed by Balakrishna, Sanyuk, Schechter and Subbaraman [6] to construct a stable chiral quantum soliton within the original chiral σ-model. Then we apply this method to derive the results for isovector exchange magnetic moment operators for two-nucleon systems in the constant-cutoff approach to the SU(2) σ-model using the product Ansatz for the soliton field operator [7]. Such an approach avoids very lengthy algebraic manipulations and complicated final results [7]. On the other hand it allows the same physical description of the isovector exchange mechanisms.

The reason why the cutoff-approach to the problem of chiral quantum soliton works is connected to the fact that the solution $F = F(r)$ which satisfies the boundary condition $F(\infty) = 0$ is singular at $r = 0$. From the physical point of view the chiral quantum model is not applicable to the region about the origin, since in that region there is a quark- dominated bag of the soliton.

However, as argued in [6], when a cutoff ϵ is introduced then the boundary conditions $F(\epsilon) = -n\pi$ and $F(\infty) = 0$, can be satisfied. In [6] an interesting analogy with the damped pendulum has been discussed, showing clearly that as long as $\epsilon > 0$, there is a chiral phase $F = F(r)$ satisfying the above boundary conditions. The asymptotic forms of such a solution are given by Eq. (2.2) in [6]. From these asymptotic solutions we immediately see that for $\epsilon \longrightarrow 0$ the chiral phase diverges at the lower limit.

Different applications of the constant-cutoff approach have been discussed in [8].

2 Constant-Cutoff Stabilization

The chiral soliton with baryon number $n = 1$ is given by (1.4), where $F = F(r)$ is the radial chiral phase function satisfying the boundary conditions $F(0) = -\pi$ and $F(\infty) = 0$.

Substituting (1.4) into (1.1) we obtain the static energy of the chiral baryon

$$M = \frac{\pi}{2} F_\pi^2 \int_{\varepsilon(t)}^{\infty} dr \left[r^2 \left(\frac{dF}{dr} \right)^2 + 2\sin^2 F \right] \ . \tag{2.1}$$

In (2.1) we avoid the singularity of the profile function $F = F(r)$ at the origin by introducing the cutoff $\epsilon(t)$ at the lower boundary of the space interval $r \in [0, \infty]$, i.e. by working with the interval $r \in [\varepsilon, \infty]$. The cutoff itself is introduced following [6] as a dynamic time-dependent variable.

From (2.1) we obtain the following differential equation for the profile function $F = F(r)$

$$\frac{d}{dr}\left(r^2 \frac{dF}{dr}\right) = \sin(2F) \ , \tag{2.2}$$

with the boundary conditions $F(\varepsilon) = -\pi$ and $F(\infty) = 0$, such that the correct soliton number is obtained. The profile function $F = F[r; \epsilon(t)]$ now depends implicitly on time t through $\epsilon(t)$. Thus in the nonlinear σ-model Lagrangian

$$L = \frac{F_\pi^2}{16} \int d^3\boldsymbol{x}\, \mathrm{Tr}\left(\partial_\mu U \partial^\mu U^+\right) \ , \tag{2.3}$$

we use the Ansätze

$$U(\boldsymbol{r}, t) = A(t) U_0(\boldsymbol{r}, t) A^+(t) \ , \quad U^+(\boldsymbol{r}, t) = A(t) U_0^+(\boldsymbol{r}, t) A^+(t) \ , \tag{2.4}$$

where

$$U_0(\boldsymbol{r}, t) = \exp\left[i\boldsymbol{\tau} \cdot \boldsymbol{r}_0 F(r; \varepsilon(t))\right] \ . \tag{2.5}$$

The static part of the Lagrangian (2.3), i.e.

$$L = \frac{F_\pi^2}{16} \int d^3\boldsymbol{x}\, \mathrm{Tr}\left(\boldsymbol{\nabla} U \cdot \boldsymbol{\nabla} U^+\right) = -M \ , \tag{2.6}$$

is equal to minus the energy M given by (2.1). The kinetic part of the Lagrangian is obtained using (2.4) with (2.5) and it is equal to

$$L = \frac{F_\pi^2}{16} \int d^3\boldsymbol{x}\, \mathrm{Tr}\left(\partial_0 U \partial_0 U^+\right) = b x^2 \mathrm{Tr}\left(\partial_0 A \partial_0 A^+\right) + c\,[\dot{x}(t)]^2 \ , \tag{2.7}$$

where

$$b = \frac{2\pi}{3} F_\pi^2 \int_1^\infty dy\, y^2 \sin^2 F \ , \quad c = \frac{2\pi}{9} F_\pi^2 \int_1^\infty dy\, y^2 \left(\frac{dF}{dy}\right)^2 y^2 \ , \tag{2.8}$$

with $x(t) = [\varepsilon(t)]^{3/2}$ and $y = r/\epsilon$. On the other hand the static energy functional (2.1) can be rewritten as

$$M = a x^{2/3} \ , \quad a = \frac{\pi}{2} F_\pi^2 \int_1^\infty dy \left[y^2 \left(\frac{dF}{dy}\right)^2 + 2\sin^2 F\right] \ . \tag{2.9}$$

Thus the total Lagrangian of the rotating soliton is given by

$$L = c\dot{x}^2 - a x^{2/3} + 2b x^2 \dot{\alpha}_\nu \dot{\alpha}^\nu \ , \tag{2.10}$$

where $\mathrm{Tr}\,(\partial_0 A \partial_0 A^+) = 2\dot{\alpha}_\nu \dot{\alpha}^\nu$ and α_ν $(\nu = 0, 1, 2, 3)$ are the collective coordinates defined as in [9]. In the limit of a time-independent cutoff $(\dot{x} \to 0)$ we can write

$$H = \frac{\partial L}{\partial \dot{\alpha}^\nu} \dot{\alpha}^\nu - L = a x^{2/3} + 2 b x^2 \dot{\alpha}_\nu \dot{\alpha}^\nu = a x^{2/3} + \frac{1}{2 b x^2} J(J+1) \; , \tag{2.11}$$

where $\left\langle \boldsymbol{J}^2 \right\rangle = J(J+1)$ is the eigenvalue of the square of the soliton laboratory angular momentum. A minimum of (2.11) with respect to the parameter x is reached at

$$x = \left[\frac{2}{3} \frac{ab}{J(J+1)} \right]^{-3/8} \quad \Rightarrow \quad \varepsilon^{-1} = \left[\frac{2}{3} \frac{ab}{J(J+1)} \right]^{1/4} . \tag{2.12}$$

The energy obtained by substituting (2.12) into (2.11) is given by

$$E = \frac{4}{3} \left[\frac{3}{2} \frac{a^3}{b} J(J+1) \right]^{1/4} . \tag{2.13}$$

This result is identical to the result obtained by Mignaco and Wulck which is easily seen if we rescale the integrals a and b in such a way that $a \to \frac{\pi}{4} F_\pi^2 a$, $b \to \frac{\pi}{4} F_\pi^2 b$ and introduce $f_\pi = 2^{-2/3} F_\pi$. However in the present approach, as shown in [6], there is a profile function $F = F(y)$ with proper soliton boundary conditions $F(1) = -\pi$ and $F(\infty) = 0$ and the integrals a, b and c in (2.9-10) exist and are shown in [6] to be $a = 0.78$ GeV^2, $b = 0.91$ GeV^2, $c = 1.46$ GeV^2 for $F_\pi = 186$ MeV.

Using (2.13) we obtain the same prediction for the mass ratio of the lowest states as Mignaco and Wulck [3] which agrees rather well with the empirical mass radio for the Δ-resonance and the nucleon. Furthermore using the calculated values for the integrals a and b we obtain the nucleon mass $M(\mathrm{N}) = 1167$ MeV which is about 25% higher than the empirical value of 939 MeV. However if we choose the pion decay constant equal to $F_\pi = 150$ MeV we obtain $a = 0.507$ GeV^2 and $b = 0.592$ GeV^2 giving the exact agreement with the empirical nucleon mass.

Finally it is of interest to know how large the constant cutoffs are for the above values of the pion-decay constant in order the check if they are in the physically acceptable ball park. Using (2.12) is is easily shown that for the nucleons ($J = \frac{1}{2}$ the cutoffs are equal to

$$\varepsilon = \begin{cases} 0.22 \text{ fm} \; , & \text{for } F_\pi = 186 \text{ MeV} \\ 0.27 \text{ fm} \; , & \text{for } F_\pi = 150 \text{ MeV} \; . \end{cases} \tag{2.14}$$

Clearly, the cutoffs have to be smaller than the nucleon size (0,72 fm), and from (2.14) we see that it is the case. It should, however, be noted that the simple Skyrme model discussed here is at variance with some physical constraints since the isoscalar charge radius (~ 0.8 fm) is identical to the baryon charge radius (~ 0.5 fm).

3 Isovector Magnetic Moment in the $B = 1$ Case

The isovector component of the nuclear electromagnetic current in the Skyrme model is a Noether current associated with the symmetry of the Skyrme Lagrangian density. The isoscalar component is, however, proportional to the topological baryon current and it is not directly related to the Skyrme Lagrangian density. Thus only the isovector curent provides a possible tool for testing the quality of the Skyrme Lagrangian.

The study of the two-nucleon system ($B = 2$), in the original Skyrme model [1], gives a good description of the isovector exchange mechanisms, e.g. the long-range tensor nucleon-nucleon interactions and spin-orbit interactions [7]. On the other hand, the Skyrme model does not provide a satisfactory description of the isoscalar exchange mechanisms, e.g. isospin-independent central interaction [7].

The isovector current is the Noether current associated with the symmetry of the Lagrangian density (1.1), without the pion mass term, under the transformation

$$U \to \exp\left(\frac{1}{2} i\varepsilon^j \tau_j\right) U \exp\left(-\frac{1}{2} i\varepsilon^j \tau_j\right) , \tag{3.1}$$

where ε^j ($j = 1, 2, 3$) is the set of three infinitesimally small Noether parameters. As $\varepsilon^j \to 0$, we obtain

$$U \to U + i\varepsilon^j \left[\frac{\tau_j}{2}, U\right] = U + \varepsilon^j \delta U_j \, . \tag{3.2}$$

The Noether current associated with the transformation (3.2) is

$$V_{j\mu} = 2\mathrm{Tr}\left(\frac{\partial \mathcal{L}}{\partial(\partial^\mu U)} \delta U_j\right) = 2i\mathrm{Tr}\left(\frac{\partial \mathcal{L}}{\partial(\partial^\mu U)} \left[\frac{\tau_j}{2}, U\right]\right) \, . \tag{3.3}$$

The third component ($j = 3$) of the vector current (3.3) is the isovector electromagnetic current given by

$$J_\mu = V_{3\mu} = -i\frac{F_\pi^2}{16} \mathrm{Tr}\left(\tau_3 U^+ \partial_\mu U + \tau_3 U \partial_\mu U^+\right) \, . \tag{3.4}$$

In the present paper we consider only the space part $\boldsymbol{J}$ of (3.4), given by

$$\boldsymbol{J} = -i\frac{F_\pi^2}{16} \mathrm{Tr}\left(\tau_3 U^+ \boldsymbol{\nabla} U + \tau_3 U \boldsymbol{\nabla} U^+\right) \, . \tag{3.5}$$

Using the rotational Ansatz $U(t) = A(t)U_0A^+(t)$ and the projection theorem [7]

$$\langle N'| A\boldsymbol{\tau}A^+ |N\rangle = -\frac{1}{3} \langle N'| \boldsymbol{\sigma}^N (\boldsymbol{\tau} \cdot \boldsymbol{\tau}^N) |N\rangle \, , \tag{3.6}$$

we obtain from (3.5)

$$\boldsymbol{J}(\boldsymbol{r}) = \frac{1}{12} \boldsymbol{\sigma} \times \boldsymbol{r}\, \tau_3 F_\pi^2 \frac{\sin^2 F}{r^2} \, . \tag{3.7}$$

The isovector magnetic moment $\boldsymbol{m}$ is obtained from the definition $\boldsymbol{J} = \boldsymbol{m} \times \boldsymbol{r}$ and is given by

$$\boldsymbol{m} = \frac{1}{12} \boldsymbol{\sigma} \tau_3 F_\pi^2 \frac{\sin^2 F}{r^2} \, . \tag{3.8}$$

The total magnetic moment is given by

$$\boldsymbol{\mu} = \frac{1}{3} \int d^3\boldsymbol{r}\, r^2 \boldsymbol{m}(r) = \frac{1}{6} \Omega \tau_3 \boldsymbol{\sigma} \, , \tag{3.9}$$

where $\Omega = b\varepsilon^3$ is the moment of inertia of the rotating soliton with b and ϵ defined by (2.8) and (2.12) respectively. From (3.9) we obtain the isovector g_s-factor

$$g_s = \frac{2M_N}{3}\Omega \ . \tag{3.10}$$

4 The Exchange Magnetic Moment Operator for $B = 2$ Soliton

In order to describe a two-nucleon system we use the product Ansatz suggested by Skyrme [1] for the $B = 2$ field operator

$$U(\boldsymbol{R}_1, \boldsymbol{R}_2; \boldsymbol{r}) = U(\boldsymbol{r} - \boldsymbol{R}_1)U(\boldsymbol{r} - \boldsymbol{R}_2) = U_1U_2 \ , \tag{4.1}$$

where $\boldsymbol{R}_1$ and $\boldsymbol{R}_2$ are the coordinates of the centers of the two solitons. The unitary operator (4.1) has the correct $B = 2$ soliton form for $\boldsymbol{R}_1 = \boldsymbol{R}_2$.

Using (4.1) we obtain the space part of the isovector current

$$\boldsymbol{J} = \boldsymbol{J}_1(\boldsymbol{r} - \boldsymbol{R}_1) + \boldsymbol{J}_2(\boldsymbol{r} - \boldsymbol{R}_2) + \boldsymbol{J}_{\mathrm{EX}}(\boldsymbol{R}_1, \boldsymbol{R}_2; \boldsymbol{r}) \ , \tag{4.2}$$

where $\boldsymbol{J}_1$ and $\boldsymbol{J}_2$ are single soliton current operators of the form

$$\boldsymbol{J}_k = -i\frac{F_\pi^2}{16}\mathrm{Tr}\left(\tau_3 U_k^+ \boldsymbol{\nabla} U_k + \tau_3 U_k \boldsymbol{\nabla} U_k^+\right) \ , \quad k = 1, 2 \ , \tag{4.3}$$

and $\boldsymbol{J}_{\mathrm{EX}}$ is an irreducible exchange current operator given by

$$\boldsymbol{J}_{\mathrm{EX}} = i\frac{F_\pi^2}{16}\mathrm{Tr}\left[U_1^+ \boldsymbol{\nabla} U_1 \left(U_2\tau_3U_2^+ - \tau_3\right) + U_2 \boldsymbol{\nabla} U_2^+ \left(U_1^+\tau_3U_1 - \tau_3\right)\right] \ . \tag{4.4}$$

The soliton fields U_1 and U_2 are rotated using the expression

$$U_k \to A_k(t)U_kA_k^+(t) = A_k(t)U_0(\boldsymbol{r} - \boldsymbol{R}_k)A_k^+(t) \ , \quad k = 1, 2 \ , \tag{4.5}$$

where U_0 is the static soliton field defined by (1.4).

Since the product Ansatz (4.1) does not possess a definite symmetry under interchange of the particle coordinates, $\boldsymbol{J}_{\mathrm{EX}}$ contains both a physical symmetric part and an unphysical antisymmetric part which has no nonzero matrix elements between any two antisymmetric two-nucleon states. The symmetric part is, on the other hand, reduced to a form which contains only Pauli spin and isospin for the two nucleons.

Using (4.5) and the projection theorem (3.6), (4.4) becomes

$$\begin{aligned} \boldsymbol{J}_{\mathrm{EX}} = \frac{F_\pi^2}{16} & \left\{ \left[\frac{\sin F_1 \cos F_1 \sin F_2 \cos F_2}{r_1 r_2} ((\boldsymbol{\sigma}^2 \cdot \boldsymbol{r}_2)\boldsymbol{\sigma}^1 - (\boldsymbol{\sigma}^1 \cdot \boldsymbol{r}_1)\boldsymbol{\sigma}^2) \right. \right. \\ & + \left(\frac{dF_1}{dr_1} - \frac{\sin F_1 \cos F_1}{r_1} \right) \frac{\sin F_2 \cos F_2}{r_1 r_2} (\boldsymbol{\sigma}^1 \cdot \boldsymbol{r}_1)(\boldsymbol{\sigma}^2 \cdot \boldsymbol{r}_2) \frac{\boldsymbol{r}_1}{r_1} \\ & \left. - \left(\frac{dF_2}{dr_2} - \frac{\sin F_2 \cos F_2}{r_2} \right) \frac{\sin F_1 \cos F_1}{r_1 r_2} (\boldsymbol{\sigma}^1 \cdot \boldsymbol{r}_1)(\boldsymbol{\sigma}^2 \cdot \boldsymbol{r}_2) \frac{\boldsymbol{r}_2}{r_2} \right] (\boldsymbol{\tau}^1 \times \boldsymbol{\tau}^2) \\ & \left. -2 \sin^2 F_1 \sin^2 F_2 \left[\frac{1}{r_1^2} (\boldsymbol{\sigma}^1 \times \boldsymbol{r}_1)\tau_3^1 + \frac{1}{r_2^2} (\boldsymbol{\sigma}^2 \times \boldsymbol{r}_2)\tau_3^2 \right] \right\} , \end{aligned} \tag{4.6}$$

where $\boldsymbol{r}_1 = \boldsymbol{r} - \boldsymbol{R}_1$, $\boldsymbol{r}_2 = \boldsymbol{r} - \boldsymbol{R}_2$, $F_1 = F(r_1)$ and $F_2 = F(r_2)$.

The magnetic dipole moment operator is defined by the following expression

$$\boldsymbol{\mu}_{\mathrm{EX}} = \frac{1}{2} \int d^3\boldsymbol{r}'\, \boldsymbol{r}' \times \boldsymbol{J}_{\mathrm{EX}}(\boldsymbol{r}' - \boldsymbol{R}_1, \boldsymbol{r}' - \boldsymbol{R}_2) , \tag{4.7}$$

where $\boldsymbol{J}_{\mathrm{EX}}$ is the exchange magnetic moment operator defined by (4.6). Introducing relative and center-of-mass coordinates

$$\boldsymbol{r} = \boldsymbol{R}_1 - \boldsymbol{R}_2 \qquad \text{and} \qquad \boldsymbol{R} = \frac{1}{2}(\boldsymbol{R}_1 + \boldsymbol{R}_2) , \tag{4.8}$$

respectively, the exchange magnetic moment operator splits into two terms as follows

$$\boldsymbol{\mu}_{\mathrm{EX}} = \boldsymbol{\mu}_r + \boldsymbol{\mu}_{\mathrm{C.M.}} , \tag{4.9}$$

where

$$\boldsymbol{\mu}_r = \frac{1}{2} \int d^3\boldsymbol{\rho}\, \boldsymbol{\rho} \times \boldsymbol{J}_{\mathrm{EX}}(\boldsymbol{\rho} - \boldsymbol{r}/2, \boldsymbol{\rho} + \boldsymbol{r}/2) , \tag{4.10}$$

$$\boldsymbol{\mu}_{\mathrm{C.M.}} = \frac{1}{2}\boldsymbol{R} \times \int d^3\boldsymbol{\rho}\, \boldsymbol{J}_{\mathrm{EX}}(\boldsymbol{\rho} - \boldsymbol{r}/2, \boldsymbol{\rho} + \boldsymbol{r}/2) . \tag{4.11}$$

In (4.10) and (4.11) the vector $\boldsymbol{\rho}$ is defined by

$$\boldsymbol{\rho} = \boldsymbol{r}' - \boldsymbol{R} . \tag{4.12}$$

In the present paper only $\boldsymbol{\mu}_r$, depending solely on the relative coordinate $\boldsymbol{r}$, is of interest.

In order to calculate $\boldsymbol{\mu}_r$ we shall perform the variable transformation $\boldsymbol{\rho} \to -\boldsymbol{\rho}$, which amounts to changing the sign of the relative coordinate $\boldsymbol{r}$ since $\boldsymbol{J}_{\mathrm{EX}}$ is odd in the variables $\boldsymbol{r}_1 = \boldsymbol{r}' - \boldsymbol{R}_1$ and $b\rho_2 = \boldsymbol{r}' - \boldsymbol{R}_2$. Furthermore we make a multipole expansion of the radial functions occuring in $\boldsymbol{\mu}_r$, using

$$f(\boldsymbol{r}_1, \boldsymbol{r}_2) = \sum_{l=0}^{\infty} f_l(r_1, r_2) P_l(\boldsymbol{r}_1 \cdot \boldsymbol{r}_2) , \tag{4.13}$$

where $P_l(x)$ are Legendre polynomials and the multipole factors $f_l(r_1, r_2)$ are defined by

$$f_l(r_1, r_2) = \frac{2l+1}{2} \int_{-1}^{+1} dz\, P_l(z) f(r_1, r_2; z) \ . \tag{4.14}$$

Thus we obtain the standard form for the exchange magnetic moment

$$\begin{aligned} \boldsymbol{\mu}_r = \frac{1}{4M_N} \Big\{ & \Big[g_1 \boldsymbol{\sigma}^1 \times \boldsymbol{\sigma}^2 + g_2 \Big(((\boldsymbol{\sigma}^1 \times \boldsymbol{\sigma}^2) \cdot \boldsymbol{r}_0)\boldsymbol{r}_0 - \frac{1}{3} \boldsymbol{\sigma}^1 \times \boldsymbol{\sigma}^2 \Big) \Big] (\boldsymbol{\tau}^1 \times \boldsymbol{\tau}^2)_3 \\ & + \Big[h_1 (\boldsymbol{\sigma}^1 - \boldsymbol{\sigma}^2) + h_2 \Big(((\boldsymbol{\sigma}^1 - \boldsymbol{\sigma}^2) \cdot \boldsymbol{r}_0)\boldsymbol{r}_0 - \frac{1}{3} (\boldsymbol{\sigma}^1 - \boldsymbol{\sigma}^2) \Big) \Big] (\boldsymbol{\tau}^1 - \boldsymbol{\tau}^2)_3 \\ & + \Big[k_1 (\boldsymbol{\sigma}^1 + \boldsymbol{\sigma}^2) + k_2 \Big(((\boldsymbol{\sigma}^1 + \boldsymbol{\sigma}^2) \cdot \boldsymbol{r}_0)\boldsymbol{r}_0 - \frac{1}{3} (\boldsymbol{\sigma}^1 + \boldsymbol{\sigma}^2) \Big) \Big] (\boldsymbol{\tau}^1 + \boldsymbol{\tau}^2)_3 \Big\} \ , \end{aligned} \tag{4.15}$$

where the radial functions g_1, g_2, h_1, h_2, k_1 and k_2 are given by

$$\begin{aligned} g_1(r) = & -\frac{2\pi M_N}{27} F_\pi^2 \int_\varepsilon^\infty d\rho\, \rho^2 \int_{-1}^{+1} dz \Big\{ \frac{\sin F_1 \cos F_1 \sin F_2 \cos F_2}{r_1 r_2} \Big(\rho^2 + \frac{1}{2} r\rho z \Big) \\ & + \frac{1}{6r_1^2} \Big(\frac{dF_1}{dr_1} - \frac{\sin F_1 \cos F_1}{r_1} \Big) \frac{\sin F_2 \cos F_2}{r_2} r^2 \rho^2 (1 - P_2(z)) \Big\} \ , \end{aligned} \tag{4.16}$$

$$\begin{aligned} g_2(r) = & -\frac{3}{2} g_1 \\ & -\frac{\pi M_N}{9} F_\pi^2 \int_\varepsilon^\infty d\rho\, \rho^4 \int_{-1}^{+1} dz \Big\{ \frac{\sin F_1 \cos F_1 \sin F_2 \cos F_2}{r_1 r_2} (1 - P_2(z)) \Big\} \ , \end{aligned} \tag{4.17}$$

$$h_1(r) = k_1(r) = -\frac{2\pi M_N}{27} F_\pi^2 \int_\varepsilon^\infty d\rho\, \rho^2 \int_{-1}^{+1} dz\, \frac{\sin^2 F_1 \sin^2 F_2}{r_1^2} \Big(\rho^2 + \frac{1}{2} r\rho z \Big) \ , \tag{4.18}$$

$$h_2(r) = k_2(r) = -\frac{3}{2} k_1 - \frac{\pi M_N}{9} F_\pi^2 \int_\varepsilon^\infty d\rho\, \rho^4 \int_{-1}^{+1} dz\, \frac{\sin^2 F_1 \sin^2 F_2}{r_1^2} (1 - P_2(z)) \ . \tag{4.19}$$

Using now the constant cutoff method, we obtain the results for the radial functions (4.16-19)

$$g_1(r) = -\frac{2\pi M_N}{27} F_\pi^2 \gamma_1(r/\varepsilon) \left[\frac{3}{2ab} J(J+1) \right]^{\frac{3}{4}} \ , \tag{4.20}$$

$$g_2(r) = -\frac{3}{2} g_1(r) - \frac{\pi M_N}{9} F_\pi^2 \gamma_2(r/\varepsilon) \left[\frac{3}{2ab} J(J+1) \right]^{\frac{3}{4}} \ , \tag{4.21}$$

$$h_1(r) = k_1(r) = -\frac{2\pi M_N}{27} F_\pi^2 \eta_1(r/\varepsilon) \left[\frac{3}{2ab} J(J+1)\right]^{\frac{3}{4}} , \tag{4.22}$$

$$h_2(r) = k_2(r) = -\frac{3}{2} k_1(r) - \frac{\pi M_N}{9} F_\pi^2 \eta_2(r/\varepsilon) \left[\frac{3}{2ab} J(J+1)\right]^{\frac{3}{4}} , \tag{4.23}$$

where a and b are dimensionless integrals defined by (2.9) and (2.8) respectively, and $\langle \boldsymbol{J}^2 \rangle = J(J+1)$ is the eigenvalue of the square of the soliton collective angular momentum. In (4.20-23) we introduced the dimensionless integrals γ_1, γ_2, η_1 and η_2, defined by

$$\gamma_1(s) = \int_1^\infty d\xi\, \xi^2 \int_{-1}^{+1} dz \left\{ \frac{\sin F_1 \cos F_1 \sin F_2 \cos F_2}{s_1 s_2} \left(\xi^2 + \frac{1}{2} s\xi z\right) + \frac{1}{6s_1^2} \left(\frac{dF_1}{ds_1} - \frac{\sin F_1 \cos F_1}{s_1}\right) \frac{\sin F_2 \cos F_2}{s_2} s^2 \xi^2 (1 - P_2(z)) \right\} , \tag{4.24}$$

$$\gamma_2(s) = \int_1^\infty d\xi\, \xi^4 \int_{-1}^{+1} dz\, \frac{\sin F_1 \cos F_1 \sin F_2 \cos F_2}{s_1 s_2} (1 - P_2(z)) , \tag{4.25}$$

$$\eta_1(s) = \int_1^\infty d\xi\, \xi^2 \int_{-1}^{+1} dz\, \frac{\sin^2 F_1 \sin^2 F_2}{s_1^2} \left(\xi^2 + \frac{1}{2} s\xi z\right) , \tag{4.26}$$

$$\eta_2(s) = \int_1^\infty d\xi\, \xi^4 \int_{-1}^{+1} dz\, \frac{\sin^2 F_1 \sin^2 F_2}{s_1^2} (1 - P_2(z)) . \tag{4.27}$$

In (4.19-22) we used the notation $F_k = F(s_k)$ $(k = 1, 2)$ where $F = F(s)$ is the chiral phase function and

$$s = \frac{r}{\varepsilon} , \qquad s_k = \frac{r_k}{\varepsilon} \quad (k = 1, 2) , \qquad \xi = \frac{\rho}{\varepsilon} . \tag{4.28}$$

As the Skyrme model reproduces the usual one-pion exchange potential at large internuclear separations, the exchange magnetic moment operator (4.9) agrees with the usual pion exchange magnetic moment operator in the asymptotic region when $r \to \infty$. The form of the magnetic moment operator in the pion physics is well known, and the relevant radial function $g_2^\pi(r)$ and $g_2^\pi(r)$ are given by

$$g_1^\pi(r) = \frac{2}{3} \frac{M_N}{m_\pi} \frac{f_{\pi NN}^2}{4\pi} (2m_\pi r - 1) \frac{e^{-2m_\pi r}}{m_\pi r} , \tag{4.29}$$

$$g_2^\pi(r) = -2 \frac{M_N}{m_\pi} \frac{f_{\pi NN}^2}{4\pi} (m_\pi r + 1) \frac{e^{-2m_\pi r}}{m_\pi r} . \tag{4.30}$$

This important consistency test of the Skyrme model remains valid even in the case of the constant-cutoff approach to the SU(2) σ-model, since in the asymptotic region when $r \rightarrow \infty$, the behaviour of our expressions (4.20-23), with the integrals (4.24-27), is the same as that of the expressions (4.29-30). This consistency was discussed in the case of the complete Skyrme model in [7]. However the present approach makes the algebraic manipulations easier and final results simpler and much easier to handle.

Finally the numerical comparison of our radial functions $g_1(r)$, $g_2(r)$, $h_1(r) = k_1(r)$ and $h_2(r) = k_2(r)$ with the corresponding radial functions obtained using the complete Skyrme model in [7] is made in Figs. 1, 2, 3 and 4, respectively, for $F_\pi = 186$ MeV and for $r > \varepsilon \sim 0.22$ fm. From Figs 1, 2, 3 and 4 we see that there is a good qualitative agreement between our radial functions and the corresponding radial functions obained using the complete Skyrme model in [7], for large and intermediate separations. The product Ansatz to describe the $B = 2$ soliton is, of course, limited to large and intermediate separations. For very small separations ($r < \varepsilon$) the comparison is not possible anyway, since our radial functions are defined only for $r \in [\varepsilon, +\infty]$.

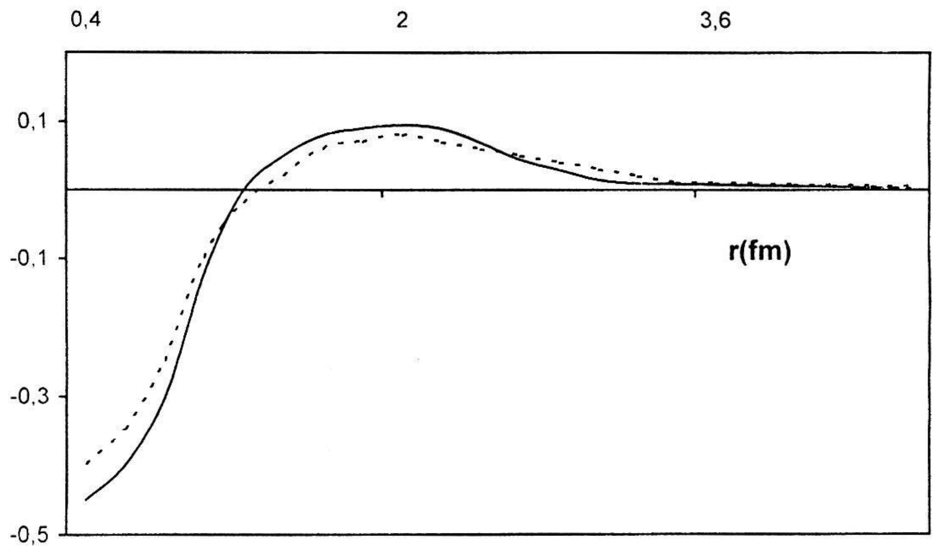

Figure 1: The radial function $g_1(r)$ given by the expression (4.20) for nucleons (solid line) compared to the corresponding radial function obtained using the complete Skyrme model [7] (dashed line).

5 Conclusions

In the present paper we have derived the expression for the isovector exchange magnetic moment in the constant-cutoff approach to the SU(2) σ-model. Thus we have shown that the long-range behaviour of the isovector exchange magnetic moment agrees with the usual

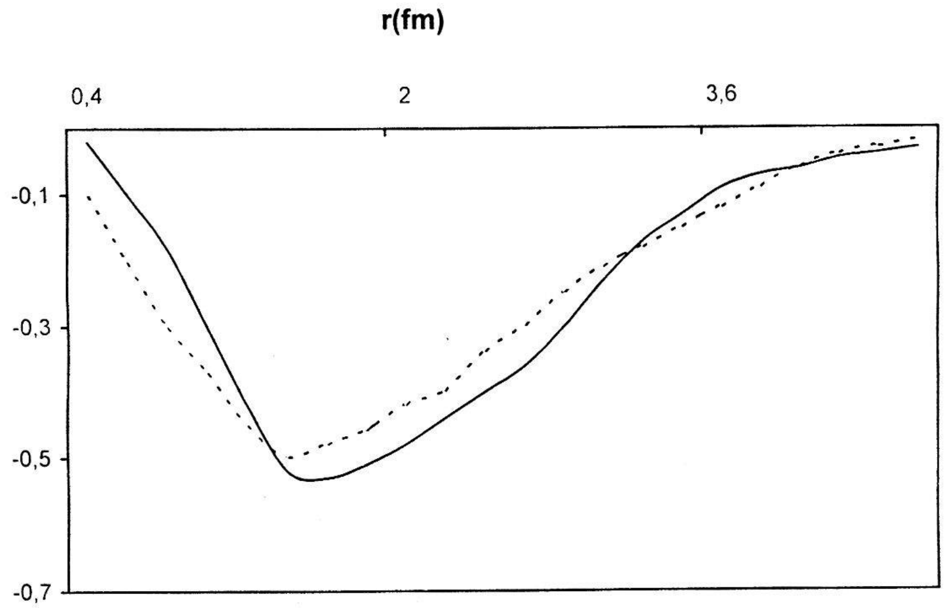

Figure 2: The radial function $g_2(r)$ given by the expression (4.21) for nucleons (solid line) compared to the corresponding radial function obtained using the complete Skyrme model [7] (dashed line).

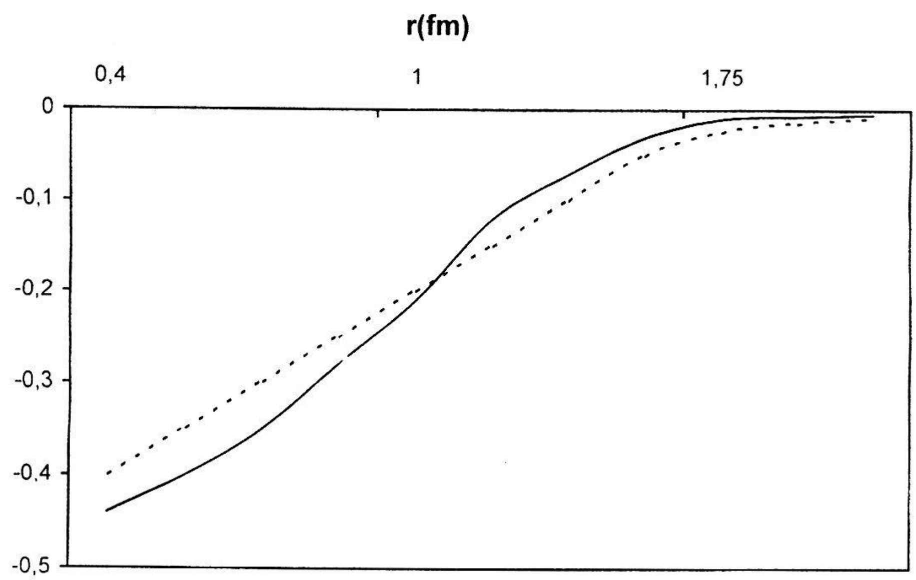

Figure 3: The radial function $h_1(r) = k_1(r)$ given by the expression (4.22) for nucleons (solid line) compared to the corresponding radial function obtained using the complete Skyrme model [7] (dashed line).

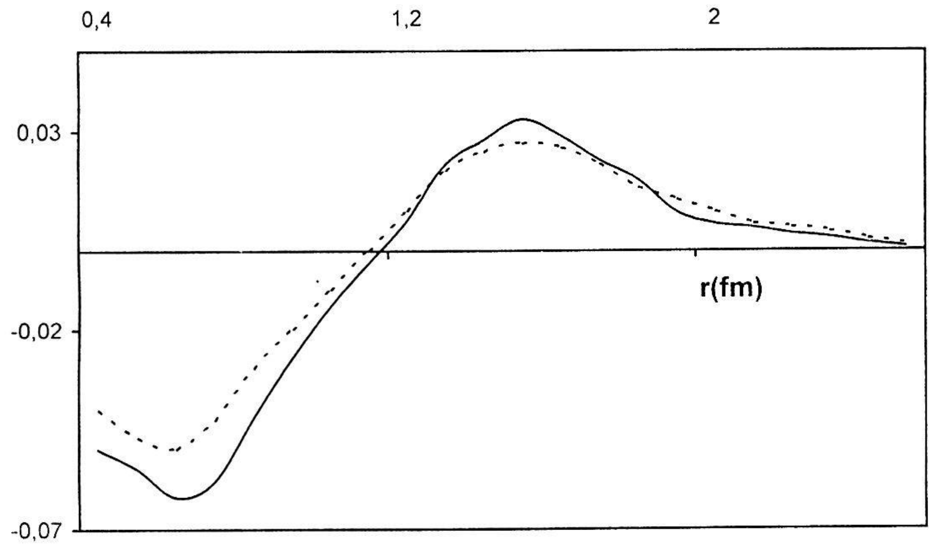

Figure 4: The radial function $h_2(r) = k_2(r)$ given by the expression (4.23) for nucleons (solid line) compared to the corresponding radial function obtained using the complete Skyrme model [7] (dashed line).

pion-exchange magnetic moment. This agreement is due to the manifest chiral invariance of the SU(2) σ-model Lagrangian without the pion mass term, which leads to predictions that satisfy the soft-pion theorems based on the chiral invariance.

We have however not investigated the short-range behaviour of the isovector magnetic moment, which is not well understtod in the complete Skyrme model (with the Skyrme stabilizing term included). We shall return to that matter in the coming studies.

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