Zeitschrift: Helvetica Physica Acta
Band: 69 (1996)
Heft: 5-6

Artikel: Analytic elements of Lie groups
Autor: Elst, A.F.M. ter / Robinson, Derek W.
DOI: https://doi.org/10.5169/seals-116972

Helv Phys Acta
Vol. 69 (1996)

0018-0238/96/060655-24$1.50+0.20/0

Analytic Elements of Lie groups

By A.F.M. ter Elst

Department of Mathematics and Comp. Sc., Eindhoven University of Technology
P.O. Box 513, 5600 MB Eindhoven, The Netherlands

and Derek W. Robinson

Centre for Mathematics and its Applications, Australian National University
Canberra, ACT 0200, Australia

(15.V.1996)

Abstract. We review the theory of C^n-, C^∞-, and analytic elements for a strongly continuous representation of a Lie group in a Banach space. We simplify some of the existing proofs and give a new, short, proof for a characterization and density of the analytic elements of a unitary representation.

1 Introduction

The distinctive feature of Lie groups is not algebraic but analytic. Each Lie group is an analytic manifold and consequently possesses a differential and analytic structure. Our purpose is to review briefly the current status of the foundations of the analytic theory and provide simplified proofs of some of the principal results.

The algebraic theory of Lie groups is largely modelled on the theory of compact groups with an emphasis on reduction theory and irreducible or factorial representations. A key role is played by certain central elements, the Casimir operators. These operators are quadratic elements in the enveloping Lie algebra which commute with the Lie algebra and are basic invariants of the theory. In contrast the analytic theory is modelled on the theory of partial differential equations for which the elliptic operators are fundamental. The simplest elliptic operators are Laplacians and their Lie analogues are again quadratic elements in the enveloping Lie algebra, the sum of squares of a basis of the Lie algebra. The Laplacians have, however, no particular invariance properties, their importance arises from properties of dissipativity and domination. In each representation the Laplacians dominate, in a precise sense, the action of the elements of the Lie algebra. Consequently they give a method of quantitatively assessing the group action. The accuracy and utility of this assessment depends on the precise measure of domination provided by the Laplacian. This in turn depends upon the nature of the representation. Unitary representations provide optimal examples and their structure will be examined in detail in Section 3.

The prime analytic observation is that each Laplacian generates a continuous semigroup, the 'heat' semigroup, in each continuous representation of the Lie group. The analytic features of the representation are largely embodied in the action of this heat semigroup. This action is determined by an integral kernel which has many properties analogous to the Gaussian kernel of the standard heat equation. Initial investigations [11] [12] of the analytic structure of Lie groups were based on detailed properties of the 'heat' kernel and much recent work has been dedicated to examination of the kernels (see, for example, [14] [17] [7]). But our description of the basic structure of the analytic theory requires no knowledge of this kernel on the Lie group.

2 Preliminaries

Let G be a Lie group, $\mathcal{X}$ a Banach space and U a representation of G by bounded operators $\{U(g) : g \in G\}$ on $\mathcal{X}$. Then U is called **strongly continuous** if the map $g \mapsto U(g)x$ from G into $\mathcal{X}$ is continuous for each $x \in \mathcal{X}$. A **unitary representation** is a strongly continuous representation in which the space $\mathcal{X} = \mathcal{H}$ is a Hilbert space and the $U(g)$ are unitary, i.e., $U(g)^* = U(g)^{-1} = U(g^{-1})$, for all $g \in G$. There are two other standard notions of continuity. The representation U is called **weakly continuous** if the map $g \mapsto (f, U(g)x)$ from G into $\mathbf{C}$ is continuous for all $x \in \mathcal{X}$ and $f \in \mathcal{X}^*$, the dual of $\mathcal{X}$. But it is a consequence of the group structure that the notions of strongly and weakly continuity coincide (see, for example, [1] Corollary 3.1.8). In the case of a unitary representation $(\mathcal{H}, G, U)$ the equivalence is a consequence of the identity

$$\|(I - U(g))x\|^2 = 2\,\mathrm{Re}\,(x, (I - U(g))x)$$

which is valid for all $x \in \mathcal{H}$ and $g \in G$. Alternatively, if $\mathcal{X} = \mathcal{Y}^*$ is the dual of a Banach space $\mathcal{Y}$, the predual, then U is called **weakly* continuous** if the adjoints $U(g)^*$ leave $\mathcal{Y}$ invariant and $g \mapsto (y, U(g)x)$ from G into $\mathbf{C}$ is continuous for all $x \in \mathcal{X}$ and $y \in \mathcal{Y}$. If the Banach space $\mathcal{X}$ is reflexive then weak* continuity is the same as weak continuity, but in general they differ. For example, the representation of $\mathbf{R}$ by translations $(U(y)f)(x) = f(x-y)$ on $L_\infty(\mathbf{R})$ is weakly* continuous, but not weakly (strongly) continuous. Since we mainly deal with representations in Hilbert spaces we will only consider strongly continuous representations. For the general theory of elliptic operators, however, weakly* continuous representations play a fundamental role.

If $a \in \mathfrak{g}$, the Lie algebra of G, then $t \mapsto U(\exp(-ta_i))$ is a continuous one parameter group of operators on $\mathcal{X}$. Here exp is the exponential map from $\mathfrak{g}$ into G. We denote the infinitesimal generator of this one parameter group by $dU(a)$, which exists by the Hille–Yosida theorem and it is a closed operator. So $U(\exp(-ta_i)) = e^{-tdU(a)}$ for all $t \in \mathbf{R}$. If the representation U is unitary then it follows from the Stone–Von Neumann theorem that the operator $dU(a)$ is skew-adjoint. For the sequel it is important to fix a basis $a_1, \ldots, a_d$ for $\mathfrak{g}$. Then we set $A_i = dU(a_i)$, for simplicity.

The Lie group is a (real) analytic manifold, so it has a C^n-, C^∞-, and an analytic structure. If $x \in \mathcal{X}$ and $n \in \mathbf{N}$ then x is called a **C^n-element**, a **C^∞-element** or an **analytic element** for U if the map $g \mapsto U(g)x$ from G into $\mathcal{X}$ is a C^n-function, a C^∞-function or a (real) analytic function, respectively. We denote the space of C^n-elements, C^∞-elements and analytic elements for U by $\mathcal{X}_n(U)$, $\mathcal{X}_\infty(U)$ and $\mathcal{X}_\omega(U)$, respectively. Occasionally we set $\mathcal{X}_0(U) = \mathcal{X}$. If no confusion is possible we write simply $\mathcal{X}_n$, $\mathcal{X}_\infty$ and $\mathcal{X}_\omega$. Since $U(h)$ is a continuous linear map from $\mathcal{X}$ into $\mathcal{X}$ and $U(hg)x = U(h)\,U(g)x$ for all $g \in G$ the element x is a C^n-element, a C^∞-element or an analytic element for U if, and only if, the map $g \mapsto U(g)x$ is a C^n-function, a C^∞-function or a (real) analytic function from a neighbourhood of the identity element e of G into $\mathcal{X}$. Using the exponential map, it is both necessary and sufficient that $a \mapsto U(\exp a)x$ is a map from a neighbourhood of $0 \in \mathfrak{g}$ into $\mathcal{X}$ with the desired regularity. Moreover, for all $h \in G$ one deduces from the identity $U(gh)x = U(g)\,U(h)x$ for all $g \in G$ that the spaces $\mathcal{X}_n(U)$, $\mathcal{X}_\infty(U)$ and $\mathcal{X}_\omega(U)$ are invariant under $U(h)$.

There is an infinitesimal description of the C^n-subspaces $\mathcal{X}_n(U)$. For this characterization it is convenient to introduce a multi-index notation. If $n \in \mathbf{N}_0$ and $\alpha = (i_1, \ldots, i_n)$ with $i_1, \ldots, i_n \in \{1, \ldots, d\}$ we write $A^\alpha = A_{i_1} \ldots A_{i_n}$ and set $|\alpha| = n$, the **length** of α. We adopt the convention $A^\alpha = I$ if $|\alpha| = 0$. Then one has the following identifications.

Lemma 2.1 *If $n \in \mathbf{N}$ then $\mathcal{X}_n(U) = \bigcap_{|\alpha| \leq n} D(A^\alpha)$. Hence $\mathcal{X}_\infty(U) = \bigcap_\alpha D(A^\alpha)$.*

Proof First, suppose $x \in \mathcal{X}_n(U)$. Let $\alpha = (i_1, \ldots, i_k)$ be a multi-index with $k \in \{0, \ldots, n\}$. Then the map $(t_1, \ldots, t_k) \mapsto U(\exp(t_1 a_{i_1}) \ldots \exp(t_k a_{i_k}))x$ is k-times continuously differentiable in a neighbourhood of $(0, \ldots, 0) \in \mathbf{R}^k$. Taking one derivative in each variable one deduces that $x \in D(A_{i_1} \ldots A_{i_k}) = D(A^\alpha)$.

The proof of the converse is by induction. We first establish the case $n = 1$. Let $x \in \bigcap_{i=1}^d D(A_i)$. Define $f: \mathbf{R}^d \to G$ by $f(t_1, \ldots, t_d) = U(\exp(t_1 a_1) \ldots \exp(t_d a_d))x$. Then it follows from the Duhamel formula that

$$f(t_1, \ldots, t_d) = x + \sum_{i=1}^d U(\exp(t_1 a_1) \ldots \exp(t_{i-1} a_{i-1})) \int_0^{t_i} ds\, U(\exp(s a_i)) A_i x$$

for all $(t_1, \ldots, t_d) \in \mathbf{R}^d$. Since U is strongly continuous it follows that f is once norm-differentiable at the origin and the differential equals $(A_1x, \ldots, A_dx)$. But then the map $(t_1, \ldots, t_d) \mapsto U(\exp(t_1a_1 + \ldots + t_da_d))x$ is also norm-differentiable at the origin, with the same differential. In particular, $\frac{d}{dt}U(\exp(ta))x|_{t=0} = dU(a)x$ exists for all $a \in \mathfrak{g}$ and the map $a \mapsto dU(a)x$ is a linear map. Since $\frac{d}{dt}U(g\exp(ta))x|_{t=0} = U(g)\,dU(a)x$ for all $g \in G$ and $a \in \mathfrak{g}$ the map $g \mapsto U(g)x$ is a C^1-function and $x \in \mathcal{X}_1(U)$. This establishes the case $n = 1$.

Next let $n \in \mathbf{N}$ and suppose $\bigcap_{|\alpha|\leq n} D(A^\alpha) \subset \mathcal{X}_n(U)$. Let $x \in \bigcap_{|\alpha|\leq n+1} D(A^\alpha)$. Then $A_ix \in \bigcap_{|\alpha|\leq n} D(A^\alpha) \subset \mathcal{X}_n(U)$ for all $i \in \{1, \ldots, d\}$. But the first order right derivative in the direction a_i of the map $g \mapsto U(g)x$ equals $g \mapsto U(g)\,A_ix$, this map is n-times differentiable in g and the derivatives are continuous. Thus $x \in \mathcal{X}_{n+1}(U)$ and the first statement of the lemma is established.

The second statement of the lemma is a direct consequence of the first. □

We define a norm $\|\cdot\|_n$ on $\mathcal{X}_n(U)$ by

$$\|x\|_n = \max_{|\alpha|\leq n} \|A^\alpha x\| \quad .$$

Since all the A_i are closed operators the space $\mathcal{X}_n(U)$ is a Banach space. There are many other equivalent norms which could be used in place of the foregoing l_∞-norm. In the subsequent discussion of unitary representations some estimates are optimized by use of an l_2-version of the norm.

We have shown that the space $\mathcal{X}_n(U)$ is invariant under U. Next we prove that the restriction of the representation of U to the space $\mathcal{X}_n(U)$ is (strongly) continuous.

Lemma 2.2 *If $n \in \mathbf{N}$ then the restriction of U to $\mathcal{X}_n(U)$ is strongly continuous.*

Proof Since the representation U is bounded on bounded subsets of G one easily deduces that for each compact subset K of G the maps $g \mapsto U(g)x$ are equicontinuous from G into $\mathcal{X}$ uniformly for all $x \in K$.

Let $k \in \{1, \ldots, n\}$ and $i_1, \ldots, i_k \in \{1, \ldots, d\}$. Then for all $x \in \mathcal{X}_n(U)$ one has the decomposition

$$\begin{aligned} &A_{i_1} \ldots A_{i_k}(U(g)x - x) \\ &\quad = \Big(U(g)\,dU(\mathrm{Ad}(g^{-1})a_{i_1}) \ldots dU(\mathrm{Ad}(g^{-1})a_{i_k})x - dU(\mathrm{Ad}(g^{-1})a_{i_1}) \ldots dU(\mathrm{Ad}(g^{-1})a_{i_k})x\Big) \\ &\qquad + \Big(dU(\mathrm{Ad}(g^{-1})a_{i_1}) \ldots dU(\mathrm{Ad}(g^{-1})a_{i_k})x - dU(a_{i_1}) \ldots dU(a_{i_k})x\Big) \end{aligned}$$

for all $g \in G$. The lemma follows from this decomposition, the above uniform continuity and some elementary estimates. □

It follows from this lemma that one has a discrete family of continuous representations $(\mathcal{X}_n, U^{(n)}, G)$ obtained by restriction of U to the C^n-subspaces and the next lemma shows that the C^n-structures are compatible.

Lemma 2.3 *If $n, m \in \mathbf{N}$ and $U^{(n)}$ denotes the restriction of U to the space $\mathcal{Y} = \mathcal{X}_n(U)$ then $\mathcal{Y}_m(U^{(n)}) = \mathcal{X}_{n+m}(U)$, with equivalent norms.*

Proof First, let $A_i^{(n)} = dU^{(n)}(a_i)$ be the infinitesimal generator with respect to the representation $U^{(n)}$. Then $A_i^{(n)} \subseteq A_i$ for all i. Let $x \in \mathcal{Y}_m(U^{(n)})$. Then $x \in \bigcap_{|\alpha| \le m} D(A^{(n)\alpha})$. Therefore for all α with $|\alpha| \le m$ one has $x \in D(A^\alpha)$ and

$$A^\alpha x = A^{(n)\alpha} x \in \mathcal{X}_n(U) = \bigcap_{|\beta| \le n} D(A^\beta) \quad .$$

Hence $x \in \bigcap_{|\gamma| \le n+m} D(A^\gamma) = \mathcal{X}_{n+m}(U)$.

Conversely, let $x \in \mathcal{X}_{n+m}(U)$. Then the map $(g, h) \mapsto U(h)\, U(g)x$ from $G \times G$ into $\mathcal{X}$ is a C^{n+m}-function. Differentiating with respect to h and evaluating at $h = e$ one deduces that the function $g \mapsto A^\alpha U(g)x$ from G into $\mathcal{X}$ is a C^m-function for all α with $|\alpha| \le n$. Hence the map $g \mapsto U^{(n)}(g)x$ from G into $\mathcal{X}_n(U) = \mathcal{Y}$ is a C^m-function and $x \in \mathcal{Y}_m(U^{(n)})$. □

There is also an infinitesimal description of analytic elements.

Lemma 2.4 *If $x \in \mathcal{X}$ then $x \in \mathcal{X}_\omega(U)$ if, and only if, $x \in D(A^\alpha)$ for all α and there exist $c, t > 0$ such that $\|A^\alpha x\| \le c\, t^{|\alpha|} |\alpha|!$ uniformly for all multi-indices α.*

Proof First assume the norm bounds on $A^\alpha x$. Then, using the inequalities $(n+m)! \le 2^{n+m} n!\, m!$, it follows that there exist $b, M > 0$ such that

$$\|A_1^{n_1} \ldots A_d^{n_d} x\| \le b\, M^{n_1 + \ldots + n_d} n_1! \ldots n_d!$$

uniformly for $n_1, \ldots, n_d \in \mathbf{N}_0^d$. Hence the series

$$\sum_{n_1, \ldots, n_d = 0}^{\infty} \frac{t_1^{n_1} \ldots t_d^{n_d}}{n_1! \ldots n_d!} A_1^{n_1} \ldots A_d^{n_d} x$$

converges for $t_1, \ldots, t_d \in \langle -M^{-1}, M^{-1} \rangle$. Therefore

$$(t_1, \ldots, t_d) \mapsto \sum_{n_1, \ldots, n_d = 0}^{\infty} \frac{t_1^{n_1} \ldots t_d^{n_d}}{n_1! \ldots n_d!} A_1^{n_1} \ldots A_d^{n_d} x = U(\exp(t_1 a_1) \ldots \exp(t_d a_d))x$$

is a real analytic function from $\langle -M^{-1}, M^{-1} \rangle^d$ into G and $x \in \mathcal{X}_\omega(U)$.

Conversely, suppose $x \in \mathcal{X}_\omega(U)$. Then $x \in \mathcal{X}_\infty(U) = \bigcap_\alpha D(A^\alpha)$. Since the map $(t_1, \ldots, t_d) \mapsto U(\exp(t_1 a_1) \ldots \exp(t_d a_d))x$ from $\mathbf{R}^d$ into $\mathcal{X}$ is (real) analytic there exist $c, t > 0$ such that

$$\|A_1^{n_1} \ldots A_d^{n_d} x\| \leq c\, t^{n_1+\ldots+n_d}(n_1 + \ldots + n_d)! \tag{2.1}$$

for all $n_1, \ldots, n_d \in \mathbf{N}_0$ where we may assume that $t \geq 2dK$, with $K = \max_{i,j,k} |c_{ij}^k|$ and c_{ij}^k the structure constants. Similar bounds follow for any other reordering of the a_i. Unfortunately this argument does not establish bounds $\|A^\alpha x\| \leq c\, t^{|\alpha|} |\alpha|!$ for all multi-indices α. Nevertheless one can deduce the bounds for general multi-indices from the bounds for ordered multi-indices, i.e., the indices $(i_1, \ldots, i_n)$ with $i_1 \leq i_2 \leq \ldots \leq i_n$, by the following argument.

Let $J_{n,m}$ be the set of all multi-indices of length n with the property that if one deletes m indices from $\alpha \in J_{n,m}$ then the remaining $n - m$ indices are ordered. We shall prove that

$$\|A^\alpha x\| \leq c\, 2^m\, t^n\, n! \qquad \text{for all } \alpha \in J_{n,m} \tag{2.2}$$

for all $n, m \in \mathbf{N}_0$ with $n \geq m$. Once this is achieved one has $\|A^\alpha x\| \leq c\,(2t)^{|\alpha|} |\alpha|!$ for each α, because $\alpha \in J_{|\alpha|,|\alpha|}$, and the proof is complete.

The proof of (2.2) is by double induction, first on n and then on m.

If $n = 0$ then (2.2) is trivial. Let $N \in \mathbf{N}$ and suppose (2.2) is valid for $n = N - 1$ and all $m \in \{0, \ldots, N-1\}$. Now for $m = 0$ and $n = N$ the bounds (2.2) are a reformulation of (2.1). So let $M \in \{1, \ldots, N\}$ and suppose that (2.2) is valid for $n = N$ and $m = M - 1$. Since $J_{N,N} = J_{N,N-1}$ we may assume that $M \leq N - 1$. Let $\alpha \in J_{N,M}$. We now commute one of the misordered indices to the correct place. Since

$$A^\beta A_{i_1} \ldots A_{i_k} A_j A^\gamma = A^\beta A_j A_{i_1} \ldots A_{i_k} A^\gamma + \sum_{l=1}^{k} A^\beta A_{i_1} \ldots A_{i_{l-1}} [A_{i_l}, A_j] A_{i_{l+1}} A_{i_k} A^\gamma$$

there exist $\alpha_0 \in J_{N,M-1}$, $\alpha_1, \ldots, \alpha_{dN} \in J_{N-1,M}$ and $c_1, \ldots, c_{dN} \in [-K, K]$ such that

$$A^\alpha = A^{\alpha_0} + \sum_{i=1}^{dN} c_i\, A^{\alpha_i} \quad .$$

Using the two induction hypotheses it then follows that

$$\begin{aligned}
\|A^\alpha x\| &\leq \|A^{\alpha_0} x\| + \sum_{i=1}^{dN} |c_i|\, \|A^{\alpha_i} x\| \\
&\leq c\, 2^{M-1}\, t^N N! + d\, N\, K\, c\, 2^M t^{N-1} (N-1)! \\
&\leq c\, 2^{M-1}\, t^N\, N! + c\, 2^{M-1}\, t^N N! = c\, 2^M t^N N!
\end{aligned}$$

and (2.2) is valid for $n = N$ and $m = M$. □

We have defined several spaces associated with a strongly continuous representation but it is not clear that these spaces are non-trivial. But the next proposition shows that it is rather easy to deduce that the space $\mathcal{X}_\infty(U)$ of C^∞-elements is dense in $\mathcal{X}$. Hence the spaces $\mathcal{X}_n(U)$ are dense in $\mathcal{X}$ for all $n \in \mathbf{N}$.

Proposition 2.5 (Gårding, [8]) *The space $\mathcal{X}_\infty(U)$ is dense in $\mathcal{X}$.*

Proof If $\varphi \in C_c^\infty(G)$ and $x \in \mathcal{X}$ then

$$U(\varphi)\, x = \int_G dg\, \varphi(g)\, U(g)x \in \mathcal{X}_\infty(U)$$

by an elementary calculation. Now let $\varphi_1, \varphi_2, \ldots \in C_c^\infty(G)$ be a bounded approximation of the identity. Then $\lim_{n\to\infty} U(\varphi_n)x = x$ in $\mathcal{X}$ for all $x \in \mathcal{X}$ and $\mathcal{X}_\infty(U)$ is dense in $\mathcal{X}$. □

Corollary 2.6 *For all $n \in \mathbf{N}$ the space $\mathcal{X}_\infty(U)$ is dense in $\mathcal{X}_n(U)$.*

Proof Let $U^{(n)}$ denote the restriction of U to the space $\mathcal{Y} = \mathcal{X}_n(U)$. Then $\mathcal{Y}_\infty(U^{(n)})$ is dense in $\mathcal{Y} = \mathcal{X}_n(U)$ by Proposition 2.5. But $\mathcal{X}_\infty(U) = \mathcal{Y}_\infty(U^{(n)})$ by Lemma 2.3. □

It follows from the proof of Proposition 2.5 that the vector space

$$\mathcal{X}_G(U) = \operatorname{span}\{U(\varphi)x : \varphi \in C_c^\infty(G),\ x \in \mathcal{X}\} \quad ,$$

which is usually called the **Gårding space**, is dense in $\mathcal{X}$ and is a subspace of $\mathcal{X}_\infty(U)$. It is a much deeper result of Dixmier and Malliavin [4] that the spaces $\mathcal{X}_G(U)$ and $\mathcal{X}_\infty(U)$ are equal.

Despite the short proof for the density of the C^∞-elements for any representation, it is much more difficult to deduce that the space $\mathcal{X}_\omega(U)$ of analytic elements for U is dense in $\mathcal{X}$. This result has been proved by Cartier–Dixmier [3], Nelson [12], Langlands [11] and Gårding [9] for any continuous representation. For a self-contained, direct proof we refer to [14] Theorem II.2.2. In the next section we give a new, short and rather easy proof for the density of the analytic elements for unitary representations. In Section 4 we explain how this proof can be extended to a general continuous representation.

3 Unitary representations

The aim of this section is to characterize the spaces of C^n-, C^∞-, and analytic elements for a unitary representation. All these spaces involve the infinitesimal generators $A_1, \ldots, A_d$, i.e., the generalized partial derivatives, and the remarkable fact is that they can be characterized by one single operator, the Laplacian.

Let U be a unitary representation of the Lie group G in a Hilbert space $\mathcal{H}$, fix a basis $a_1, \ldots, a_d$ for the Lie algebra $\mathfrak{g}$ of G and set $A_i = dU(a_i)$. The Laplacian is initially defined as $\Delta = -\sum_{i=1}^d A_i^2$ with domain $D(\Delta) = \bigcap_{i=1}^d D(A_i^2)$, the space of all separately twice differentiable elements. One readily checks that Δ is a positive, symmetric, operator but it is not evident that it is closed or self-adjoint and one aim of the subsequent analysis is to establish these properties. For this purpose it is useful to consider two other possible definitions of the Laplacian involving different domains.

The largest natural domain occurs with the quadratic form definition. Let δ denote the sesquilinear form $\delta: \mathcal{H}_1 \times \mathcal{H}_1 \to \mathbf{C}$ with values

$$\delta(y, x) = \sum_{i=1}^d (A_i y, A_i x) \quad .$$

Then δ is positive, symmetric, densely defined and closed. Hence it automatically determines a positive, self-adjoint, operator which we denote by Δ_δ (see [10] Chapter VI). The domain $D(\Delta_\delta)$ of Δ_δ consists of those $x \in \mathcal{H}_1$ for which there is a $z \in \mathcal{H}$ such that

$$\delta(y, x) = (y, z)$$

for all $y \in \mathcal{H}_1$. Then the action of Δ_δ is given by $\Delta_\delta x = z$. It follows straightforwardly that Δ_δ is an extension, the self-adjoint form extension, of Δ. The advantage of this definition lies in the self-adjointness of Δ_δ which gives access to spectral theory, functional calculus, etc., but the disadvantage is that the domain of Δ_δ is only specified in an implicit manner.

The smallest natural domain for the Laplacian is the subspace of C^∞-elements and we denote by Δ_∞ the restriction of Δ to this domain, i.e., $D(\Delta_\infty) = \mathcal{H}_\infty$. Then Δ_∞ is symmetric, and hence closable, but it is not closed in general.

One of the main conclusions of this section will be that the operator Δ equals Δ_δ. Hence Δ is closed and self-adjoint on the domain $D(\Delta)$. Moreover, this explicitly identifies the form domain $D(\Delta_\delta)$ with $D(\Delta)$. In addition we prove that Δ is equal to the closure of Δ_∞. Thus the C^∞-elements $\mathcal{H}_\infty$ are a core of Δ.

It is convenient for the subsequent discussion to introduce a fourth Laplacian Δ_2 as the restriction of Δ to the subspace $\mathcal{H}_2$ of all jointly twice differentiable elements. Since $\mathcal{H}_\infty$ is dense in $\mathcal{H}_2$ by Corollary 2.6 it follows that the closures of Δ_∞ and Δ_2 coincide.

Set $M = (\sum_{i,j,k=1}^d |c_{ij}^k|^2)^{1/2}$, where c_{ij}^k are the structure constants of the Lie algebra with respect to the basis $a_1, \ldots, a_d$.

Proposition 3.1 *If* $\varepsilon > 0$ *then*

$$\sum_{i=1}^d \|A_i x\|^2 = (x, \Delta_2 x) \leq \varepsilon \|\Delta_2 x\|^2 + (4\varepsilon)^{-1} \|x\|^2 \tag{3.1}$$

and

$$\sum_{i,j=1}^{d} \|A_i A_j x\|^2 \leq (1+\varepsilon)\|\Delta_2 x\|^2 + 2M^4(1+\varepsilon)^4\varepsilon^{-3}\|x\|^2$$

uniformly for all $x \in \mathcal{H}_2$.

Proof The identity in (3.1) follows because the A_i are skew-symmetric and the bounds because

$$|(x, \Delta_2 x)| \leq \|x\| \cdot \|\Delta_2 x\| \leq \varepsilon\|\Delta_2 x\|^2 + (4\varepsilon)^{-1}\|x\|^2$$

for all $\varepsilon > 0$. Next set $U_1(x) = (\sum_{i=1}^{d} \|A_i x\|^2)^{1/2}$ and $U_2(x) = (\sum_{i,j=1}^{d} \|A_i A_j x\|^2)^{1/2}$. Then

$$\begin{aligned} U_2(x)^2 &= -\sum_{i,j=1}^{d} (A_j x, A_j A_i^2 x) - \sum_{i,j=1}^{d} (A_j x, [A_i^2, A_j] x) \\ &= \sum_{i,j=1}^{d} (A_j^2 x, A_i^2 x) - \sum_{i,j,k=1}^{d} c_{ij}^k \Big((A_j x, A_i A_k x) + (A_j x, A_k A_i x) \Big) \end{aligned}$$

for all $x \in \mathcal{H}_\infty$. Therefore,

$$\begin{aligned} U_2(x)^2 &\leq \|\Delta_2 x\|^2 + \Big| \sum_{i,j,k=1}^{d} c_{ij}^k \Big((A_j x, A_i A_k x) + (A_j x, A_k A_i x) \Big) \Big| \\ &\leq \|\Delta_2 x\|^2 + 2M\, U_1(x)\, U_2(x) \qquad (3.2) \\ &\leq \|\Delta_2 x\|^2 + \varepsilon(1+\varepsilon)^{-1} U_2(x)^2 + M^2(1+\varepsilon)\varepsilon^{-1} U_1(x)^2 \end{aligned}$$

for all $\varepsilon > 0$ and $x \in \mathcal{H}_\infty$ by use of the Cauchy–Schwarz inequality. But this can be solved to yield

$$U_2(x)^2 \leq (1+\varepsilon)\|\Delta_2 x\|^2 + M^2(1+\varepsilon)^2\varepsilon^{-1} U_1(x)^2$$

and then using the first statement one finds

$$U_2(x)^2 \leq (1+2\varepsilon)\|\Delta_2 x\|^2 + 4^{-1}M^4(1+\varepsilon)^4\varepsilon^{-3}\|x\|^2 \quad .$$

Finally replacing ε by $\varepsilon/2$ one obtains the second statement for all $x \in \mathcal{H}_\infty$ and then by closure for all $x \in \mathcal{H}_2$. □

These estimates establish that the seminorm $x \mapsto U_2(x)$ is relatively bounded by the seminorm $x \mapsto \|\Delta_2 x\|$ on $\mathcal{H}_2$ with relative bound one. Obviously, $\|\Delta_2 x\|^2 \leq d^{1/2} U_2(x)$ for all $x \in \mathcal{H}_2$. Hence the norms $x \mapsto \|\Delta_2 x\| + \|x\|$ and $x \mapsto U_2(x) + \|x\|$ are equivalent with the norm $\|\cdot\|_2$, the C^2-norm on $\mathcal{H}_2$. But since $\mathcal{H}_2$ is complete one has the following conclusion.

Corollary 3.2 *The operator* Δ_2 *is closed.*

Much more is true. The operator Δ_2 is in fact self-adjoint on $\mathcal{H}_2$. This property is the most critical element in the analysis of the differential structure of the representations and its proof requires techniques from the classical theory of elliptic differential operators. The basis of the proof is the exponential map which gives a local diffeomorphism from $\mathbf{R}^d$ to G.

Theorem 3.3 *The operator Δ_2 is self-adjoint.*

Proof Since Δ_2 is closed, Corollary 3.2, and symmetric it suffices to prove that the range of $\lambda I + \Delta_2$ is equal to $\mathcal{H}$ for some $\lambda > 0$. This relies on comparison with the $\mathbf{R}^d$-theory. There are various ways of accomplishing this but we use a parametrix argument which requires no deep knowledge of the properties of elliptic operators.

Let $\Omega \subset G$ be an open relatively compact neighbourhood of the identity $e \in G$ and W_0 an open ball in $\mathfrak{g}$ centred at the origin such that $\exp|_{W_0} : W_0 \to \Omega$ is an analytic diffeomorphism. Set $a_x = \sum_{i=1}^d x_i a_i$, for $x \in \mathbf{R}^d$, and $W = \{x \in \mathbf{R}^d : a_x \in W_0\}$. Then for $\varphi: \Omega \to \mathbf{C}$ define $\tilde{\varphi}: W \to \mathbf{C}$ by $\tilde{\varphi}(x) = \varphi(\exp(a_x))$. If Ω is small enough the image of Haar measure under this map is absolutely continuous with respect to Lebesgue measure. In particular, there exists a positive C^∞-function σ on W, bounded from below by a strictly positive constant, with all derivatives bounded on W and such that

$$\int_\Omega dg\, \varphi(g) = \int_W dx\, \sigma(x)\, \tilde{\varphi}(x)$$

for all $\varphi \in L_1(\Omega\,; dg)$. We normalize the Haar measure dg such that $\sigma(0) = 1$.

On G one has the generators B_i of left translations $(B_i\varphi)(g) = \frac{d}{dt}\varphi(\exp(-ta_i)\, g)|_{t=0}$. The key feature of the exponential map is to transpose these vector fields to C^∞-vector fields $X_1, \ldots, X_d$ on W with the property

$$(X_i\tilde{\varphi})(x) = (\widetilde{B_i\varphi})(x) = (B_i\varphi)(\exp(a_x)) = \frac{d}{dt}\varphi(\exp(-ta_i)\exp(a_x))\Big|_{t=0}$$

for all $\varphi \in C_c^\infty(\Omega)$. Moreover,

$$X_i\tilde{\varphi} = -\partial_i\tilde{\varphi} + Y_i\tilde{\varphi}$$

for all $\varphi \in C_c^\infty(\Omega)$ where the Y_i are C^∞-vector fields of the form $Y_i = \sum_{j=1}^d f_j\partial_j$ and the $f_j \in C^\infty(W)$ have a first-order zero at the origin. But then

$$(\Delta_B\varphi)^\sim = \widetilde{\Delta}\,\tilde{\varphi} + \widetilde{H}'\tilde{\varphi}$$

where $\Delta_B = -\sum_{i=1}^d B_i^2$ is the Laplacian on G, $\widetilde{\Delta} = -\sum_{i=1}^d \partial_i^2$ is the ordinary Laplacian on $\mathbf{R}^d$ and $\widetilde{H}'$ is an operator of the form

$$\widetilde{H}' = \sum_{i,j=1}^d f_{ij}\,\partial_i\,\partial_j + \sum_{i=1}^d f_i\,\partial_i + f_0$$

with $f_{ij}, f_i, f_0 \in C^\infty(W)$ and $f_{ij}(0) = 0$.

Let $\chi, \chi' \in C_c^\infty(G)$, $\operatorname{supp}\chi' \subset \Omega$, $\chi(e) = 1$ and $\chi' = 1$ on $\operatorname{supp}\chi$. Then for all $\xi \in \mathcal{H}_\infty$ and $\eta \in \mathcal{H}$ one has for all $r \in C_c^\infty(G)$ with $\operatorname{supp} r \subseteq \operatorname{supp}\chi$

$$\begin{aligned}\int_G dg\, r(g)\, (\eta, (\lambda I + \Delta_2) U(g)\xi) &= (\eta, (\lambda I + \Delta_2)\, U(r)\xi)\\ &= \int_G dg\, ((\lambda I + \Delta_B) r)(g)\, (\eta, U(g)\xi)\, \chi'(g)\\ &= \int_G dg\, r(g)\, ((\lambda I + \Delta_B)\tau)(g) \qquad (3.3)\end{aligned}$$

where $\tau(g) = (\eta, U(g)\xi)\, \chi'(g)$. Since $C_c^\infty(G)$ is dense in $L_1(G)$ it follows by continuity that (3.3) is valid for all $r \in L_1(G)$ with $\operatorname{supp} r \subseteq \operatorname{supp}\chi$. Now let r_λ be the function on G with support contained in Ω such that $\tilde{r}_\lambda = \check{R}_\lambda \tilde{\chi}$ where $\check{R}_\lambda$ denotes the kernel of the resolvent of $(\lambda I + \widetilde{\Delta})^{-1}$ on $\mathbf{R}^d$. Then

$$\begin{aligned}(\eta, (\lambda I + \Delta_2) U(r_\lambda)\xi) &= \int_W dx\, \sigma(x)\, \check{R}_\lambda(x)\, \tilde{\chi}(x)\, ((\lambda I + \widetilde{\Delta} + \widetilde{H}')\tilde{\tau})(x)\\ &= \int_W dx\, \sigma(x)\, ((\lambda I + \widetilde{\Delta} + \widetilde{H}')(\check{R}_\lambda \tilde{\chi}))(x)\, \tilde{\tau}(x)\\ &= \int_W dx\, \sigma(x)\, \delta(x)\, \tilde{\chi}(x)\, \tilde{\tau}(x) + \int_W dx\, \sigma(x)\, \tilde{s}_\lambda(x)\, (\eta, U(\exp(a_x))\xi) \quad ,\end{aligned}$$

in the sense of distributions, where $\tilde{s}_\lambda$ has the form

$$\tilde{s}_\lambda(x) = \sum_k (L^{(k)} \check{R}_\lambda)(x)\, \tilde{\chi}_k(x)\, \tilde{\chi}'(x) \quad .$$

Here the $\tilde{\chi}_k \in C^\infty(W)$ and the $L^{(k)}$ are operators of the same form as $\widetilde{H}'$ with coefficients $f_{ij}^{(k)}$, etc., in $C^\infty(W)$ and with $f_{ij}^{(k)}(0) = 0$.

Now $\check{R}_\lambda(x) = \int_0^\infty dt\, e^{-\lambda t} K_t(x)$, where $K_t(x) = (4\pi t)^{-d/2} e^{-x^2/(4t)}$ is the Gaussian. So $\|\check{R}_\lambda\|_1 = \lambda^{-1}$ and $\|r_\lambda\|_1 \le a\, \lambda^{-1/2}$. Since $|(\partial_i \partial_j K_t)(x)| \le a\, t^{-(d+2)/2} e^{-x^2/(5t)}$ and $|f_{ij}^{(k)}(x)| \le c\,|x|$, for suitable $a, c > 0$, if W is small enough, it follows that

$$\begin{aligned}\|f_{ij}^{(k)}\, \partial_i\, \partial_j\, \check{R}_\lambda\|_1 &\le \int_W dx\, c \int_0^\infty dt\, e^{-\lambda t} a\, t^{-(d+1)/2} (|x|^2 t^{-1})^{1/2} e^{-x^2/(5t)}\\ &\le a' \int_0^\infty dt\, e^{-\lambda t} t^{-(d+1)/2} \int_{\mathbf{R}^d} dx\, e^{-x^2/(6t)} \le a''\, \lambda^{-1/2}\end{aligned}$$

for all $\lambda > 0$, for suitable $a', a'' > 0$. Similarly, one can estimate the contributions of the other terms in $L^{(k)}$ and deduce that $\|L^{(k)} \check{R}_\lambda\|_1 \le a\, \lambda^{-1/2}$ for some $a > 0$, uniformly for all $\lambda \ge 1$. Hence $\|s_\lambda\|_1 \le a\, \lambda^{-1/2}$ for some $a > 0$, uniformly for all $\lambda \ge 1$. So

$$(\eta, (\lambda I + \Delta_2) U(r_\lambda)\xi) = (\eta, \xi) + (\eta, U(s_\lambda)\xi)$$

Therefore, if $R_\lambda = U(r_\lambda)$ and $S_\lambda = U(s_\lambda)$ then $\|R_\lambda \xi\| \le a\, \lambda^{-1} \|\xi\|$ and $\|S_\lambda \xi\| \le a\, \lambda^{-1/2} \|\xi\|$. Hence

$$(\lambda I + \Delta_2) R_\lambda \xi = \xi + S_\lambda \xi \qquad (3.4)$$

for all $\xi \in \mathcal{H}_\infty$. By density it follows that $R_\lambda \mathcal{H} \subseteq D(\overline{\Delta_2}) = D(\Delta_2)$ and (3.4) is valid for all $\xi \in \mathcal{H}$. Thus if $a\,\lambda^{-1/2} < 1$ then $(I + S_\lambda)$ has a bounded inverse and

$$\xi = (\lambda I + \Delta_2) R_\lambda (I + S_\lambda)^{-1} \xi \quad .$$

This establishes that the range of $(\lambda I + \Delta_2)$ is equal to $\mathcal{H}$. Hence Δ_2 is self-adjoint. □

Since a self-adjoint operator has no proper symmetric extension it immediately follows that $\Delta_2 = \Delta_\delta$. Therefore one has the following characterization of the Laplacian.

Corollary 3.4 *The Laplacian Δ is self-adjoint and $\Delta = \Delta_2 = \Delta_\delta$. The C^∞-elements $\mathcal{H}_\infty$ are a core of Δ and*

$$\mathcal{H}_2 = \bigcap_{i=1}^{d} D(A_i^2) \quad .$$

Proof Originally $\Delta_2 \subseteq \Delta \subseteq \Delta_\delta$. Then since Δ_2 and Δ_δ are both self-adjoint all three Laplacians must be equal. But $\mathcal{H}_\infty$ is clearly a core of Δ_2 and hence of Δ. Finally the equality means that $\mathcal{H}_2 = D(\Delta_2) = D(\Delta)$. □

The last statement of the corollary is rather striking as it establishes that

$$\bigcap_{i,j=1}^{d} D(A_i A_j) = \bigcap_{i=1}^{d} D(A_i^2) \quad ,$$

i.e., an element of the Hilbert space is jointly twice differentiable if, and only if, it is separately twice differentiable.

Next we consider the characterization of the n-times differentiable elements $\mathcal{H}_n$. First we begin by remarking that as Δ is now known to be self-adjoint the estimates of Proposition 3.1 can be rephrased in terms of the resolvent of Δ.

Corollary 3.5 *The operators $(\lambda I + \Delta)^{1/2}$, $\lambda > 0$, are bijections from $\mathcal{H}_1$ to $\mathcal{H}$ and from $\mathcal{H}_2$ to $\mathcal{H}_1$. Moreover,*

$$\sum_{i=1}^{d} \|A_i (\lambda I + \Delta)^{-1/2} x\|^2 \leq \|x\|^2$$

and

$$\sum_{i,j=1}^{d} \|A_i A_j (\lambda I + \Delta)^{-1} x\|^2 \leq (1 + 2M\lambda^{-1/2})^2 \|x\|^2$$

for all $x \in \mathcal{H}$ and $\lambda > 0$.

Proof As $\Delta = \Delta_\delta$ it follows from the identification of the form domain as the domain of the square root of the associated positive self-adjoint operator that $D(\delta) = D(\Delta^{1/2}) = D((\lambda I + \Delta)^{1/2})$. Therefore $D((\lambda I + \Delta)^{1/2}) = \mathcal{H}_1$ and $(\lambda I + \Delta)^{1/2}\mathcal{H}_1 = \mathcal{H}$ for all $\lambda > 0$. Moreover, as $\Delta = \Delta_2$ it follows that $D(\Delta_2) = D(\Delta) = D(\lambda I + \Delta)$ and $(\lambda I + \Delta)\mathcal{H}_2 = \mathcal{H}$ for all $\lambda > 0$. But then $(\lambda I + \Delta)^{1/2}\mathcal{H}_2 = (\lambda I + \Delta)^{-1/2}\mathcal{H} = \mathcal{H}_1$.

Next

$$\sum_{i=1}^{d} \|A_i(\lambda I + \Delta)^{-1/2}x\|^2 = (x, \Delta(\lambda I + \Delta)^{-1}x) \leq \|x\|^2$$

because Δ is positive. Moreover,

$$\sum_{i=1}^{d} \|A_i(\lambda I + \Delta)^{-1}x\|^2 = (x, \Delta(\lambda I + \Delta)^{-2}x) \leq \lambda^{-1}\|x\|^2 \quad .$$

Hence

$$\begin{aligned}\sum_{i,j=1}^{d}\|A_iA_j(\lambda I + \Delta)^{-1}x\|^2 \;\leq\;& \|\Delta(\lambda I + \Delta)^{-1}x\|^2 \\ &+ 2M\Big(\sum_{i=1}^{d}\|A_i(\lambda I + \Delta)^{-1}x\|^2\Big)^{1/2} \cdot \Big(\sum_{i,j=1}^{d}\|A_iA_j(\lambda I + \Delta)^{-1}x\|^2\Big)^{1/2}\end{aligned}$$

by the estimate (3.2). Therefore

$$\Big(\Big(\sum_{i,j=1}^{d} \|A_iA_j(\lambda I + \Delta)^{-1}x\|^2\Big)^{1/2} - M\lambda^{-1/2}\|x\|\Big)^2 \leq (1 + M^2\lambda^{-1})\|x\|^2$$

and the last statement of the corollary follows straightforwardly. □

It is an easy consequence of this last corollary that $\|\cdot\|_1$ is equivalent to the graph norm $x \mapsto \|(\lambda I + \Delta)^{1/2}x\|$ on $\mathcal{H}_1$ and $\|\cdot\|_2$ is equivalent to the norm $x \mapsto \|(\lambda I + \Delta)x\|$ on $\mathcal{H}_2$ for all $\lambda > 0$. But these results are just the simplest cases of the following characterization of the C^n-subspaces.

Theorem 3.6 *If $n \in \mathbf{N}$ then $D((I + \Delta)^{n/2}) = \mathcal{H}_n$ and the norms $x \mapsto \|(I + \Delta)^{n/2}x\|$ and $\|\cdot\|_n$ are equivalent. In particular $\bigcap_{n=1}^{\infty} D(\Delta^n) = \mathcal{H}_\infty$.*

One has $D((I + \Delta)^{1/2}) = \mathcal{H}_1$ and $D(I + \Delta) = \mathcal{H}_2$ by the foregoing discussion. The general case can then be established by induction. Basically one needs to prove the operators $A^\alpha(\lambda I + \Delta)^{-n/2}$ are bounded for all α with $|\alpha| = n$. In order to do this inductively it is necessary to commute the A_i with the resolvents. This can be done with the aid of the structure relations

$$[A_i, A_j]x = \sum_{i,j=1}^{d} c_{ij}^k A_k x \quad ,$$

which are valid for all $x \in \mathcal{H}_2$. But care has to be taken with the domains.

Lemma 3.7 *For all $i, j, k \in \{1, \ldots, d\}$ there exists $d^i_{jk} \in \mathbf{R}$ such that*

$$A_i(I+\Delta)^{-1}x = (I+\Delta)^{-1}A_i x + \sum_{j,k=1}^{d} d^i_{jk}(I+\Delta)^{-1}A_j A_k (I+\Delta)^{-1}x \tag{3.5}$$

for all $x \in \mathcal{H}_1$ and $i \in \{1, \ldots, d\}$.

Proof Let $d^i_{jk} = c^k_{ji} + c^j_{ki}$. Then

$$[\Delta, A_i]y = \sum_{j,k=1}^{d} d^i_{jk} A_j A_k y$$

for all $y \in \mathcal{H}_3$. Moreover,

$$\begin{aligned}(z, A_i(I+\Delta)^{-1}(I+\Delta)y) &= (z, (I+\Delta)^{-1}A_i(I+\Delta)y) + (z, (I+\Delta)^{-1}[(I+\Delta), A_i]y) \\ &= -(A_i(I+\Delta)^{-1}z, (I+\Delta)y) + \sum_{j,k=1}^{d} d^i_{jk}(A_k A_j (I+\Delta)^{-1}z, y)\end{aligned}$$

for all $y \in \mathcal{H}_3$ and $z \in \mathcal{H}$, because the operators $A_i(I+\Delta)^{-1}$ and $A_k A_j (I+\Delta)^{-1}$ are bounded. Since $\mathcal{H}_3$ is a core for Δ and $I+\Delta$ is a bijection from $\mathcal{H}_2$ onto $\mathcal{H}$ it follows that

$$(z, A_i(I+\Delta)^{-1}x) = -(A_i(I+\Delta)^{-1}z, x) + \sum_{j,k=1}^{d} d^i_{jk}(A_k A_j (I+\Delta)^{-1}z, (I+\Delta)^{-1}x)$$

for all $x \in \mathcal{H}$. Hence for $x \in \mathcal{H}_1$ the desired result follows by taking adjoints. □

The commutation property of the lemma immediately allows one to deduce that the resolvent improves differentiability properties by two units.

Lemma 3.8 *If $n \in \mathbf{N}_0$ then $(I+\Delta)^{-1}\mathcal{H}_n = \mathcal{H}_{n+2}$.*

Proof If $x \in \mathcal{H}_{n+2}$ then $y = (I+\Delta)x \in \mathcal{H}_n$ and $x = (I+\Delta)^{-1}y$. Hence $\mathcal{H}_{n+2} \subseteq (I+\Delta)^{-1}\mathcal{H}_n$.

The proof of the converse inclusion is by induction. The case $n = 0$ follows from Corollary 3.5. Let $n \in \mathbf{N}_0$ and suppose that $(I+\Delta)^{-1}\mathcal{H}_n \subseteq \mathcal{H}_{n+2}$. Then for $x \in \mathcal{H}_{n+1} \subseteq \mathcal{H}_1$ one has $A_i x \in \mathcal{H}_n$ and $(I+\Delta)^{-1}A_i x \in \mathcal{H}_{n+2}$ by the induction hypothesis. Moreover, $(I+\Delta)^{-1}x \in \mathcal{H}_{n+2}$ and $A_j A_k (I+\Delta)^{-1}x \in \mathcal{H}_n$. Hence $(I+\Delta)^{-1}A_j A_k (I+\Delta)^{-1}x \in \mathcal{H}_{n+2}$, where we have again used the induction hypothesis. Therefore the right hand side of (3.5) is in $\mathcal{H}_{n+2}$. Hence $A_i(I+\Delta)^{-1}x \in \mathcal{H}_{n+2}$ and $(I+\Delta)^{-1}x \in \mathcal{H}_{n+3}$. □

Now the proof of the theorem is immediate.

Proof of Theorem 3.6 Since $\mathcal{H}_1 = D((I+\Delta)^{1/2}) = (I+\Delta)^{-1/2}\mathcal{H}$ by Corollary 3.5 it follows by induction from Lemma 3.8 that $\mathcal{H}_n = (I+\Delta)^{-n/2}\mathcal{H}$ for all even and odd $n \in \mathbf{N}$. But $(I+\Delta)^{-n/2}\mathcal{H} = D((I+\Delta)^{n/2})$.

Finally the equivalence of the norms is a consequence of the closed graph theorem, since the Banach spaces $D((I+\Delta)^{n/2})$ and $\mathcal{H}_n$ are both continuously embedded in $\mathcal{H}$. □

Next we compare the analytic elements $\mathcal{H}_\omega(U)$ of the representation with the analytic elements $\mathcal{H}_\omega(\Delta^{1/2})$ of the operator $\Delta^{1/2}$. In general, if T is an operator in a Banach space $\mathcal{X}$ then the space $\mathcal{X}_\omega(T)$ of **analytic elements** for T is defined as the set of all $x \in \bigcap_{n\in\mathbf{N}} D(T^n)$ for which there exist $c, t > 0$ such that $\|T^n x\| \leq c\,t^n n!$ for all $n \in \mathbf{N}_0$. Our aim is to prove the following.

Theorem 3.9 *If $(\mathcal{H}, G, U)$ is a unitary representation then*

$$\mathcal{H}_\omega(U) = \mathcal{H}_\omega(\Delta^{1/2})$$

and the subspace $\mathcal{H}_\omega(U)$ of analytic elements is dense in $\mathcal{H}$.

Proof The inclusion $\mathcal{H}_\omega(U) \subseteq \mathcal{H}_\omega(\Delta^{1/2})$ is straightforward since $\|\Delta^{n/2}x\| = (x, \Delta^n x)^{1/2} \leq d^{n/2}\|x\|_n$ for all $n \in \mathbf{N}_0$ by the triangle inequality. The converse is more difficult.

It is readily verified that $\mathcal{H}_\omega(\Delta^{1/2}) = \mathcal{H}_\omega((I+\Delta)^{1/2})$ and so it suffices to prove that $\mathcal{H}_\omega((I+\Delta)^{1/2}) \subseteq \mathcal{H}_\omega(U)$.

Set $H = I + \Delta$. Obviously $\mathcal{H}_\omega(H^{1/2}) \subseteq \bigcap_{n\in\mathbf{N}} D(\Delta^n) = \mathcal{H}_\infty$ by Theorem 3.6. Next introduce the functions $M_{n,m}$ on $\mathcal{H}_\infty$ with values

$$M_{n,m}(x) = \sup_{\alpha;\, |\alpha|=n} \|HA^\alpha H^m x\|$$

for all $m, n \in \mathbf{N}_0$. One has

$$M_{n,m}(x) \leq \sup_{\alpha;\, |\alpha|=n} \|A^\alpha H^{m+1}x\| + \sup_{\alpha;\, |\alpha|=n} \|[H, A^\alpha]H^m x\| \tag{3.6}$$

for all $n \in \mathbf{N}_0$. But it follows from Corollary 3.5 that

$$\|x\|_1 \leq \|H^{1/2}x\| \qquad \text{and} \qquad \|x\|_2 \leq (1+2M)\|Hx\|$$

for all $x \in \mathcal{X}_\infty$. Hence

$$\sup_{1\leq i\leq d} \|A_i H^{m+1}x\| \leq \|H^{m+3/2}x\|$$

and

$$\sup_{\alpha;\, |\alpha|=n} \|A^\alpha H^{m+1}x\| \leq (1+2M) \sup_{\beta;\, |\beta|=n-2} \|HA^\beta H^{m+1}x\| = (1+2M)M_{n-2,m+1}(x) \tag{3.7}$$

for $n \geq 2$ and $m \geq 0$. Moreover, the commutator in the second term on the right hand side of (3.6) can be expressed as

$$[H, A^\alpha] = \sum_{i=1}^{d} \Big(A_i [A_i, A^\alpha] + [A_i, A^\alpha] A_i \Big)$$

and each term in the sum can be evaluated with the aid of the structure relations as a linear combination of at most $d\,n$ products A^γ with $|\gamma| = n+1$. Therefore one has bounds

$$\begin{aligned} \sup_{\alpha;\, |\alpha|=n} \|[H, A^\alpha] H^m x\| &\leq 2\,d^2 M\,n \sup_{\gamma;\, |\gamma|=n+1} \|A^\gamma H^m x\| \\ &\leq 2\,d^2 M\,(1+2M)\,n\,M_{n-1,m}(x) \quad , \end{aligned}$$

for $n \geq 1$ and $m \geq 0$ where the last bound uses (3.7). Combination of these estimates then yields

$$M_{1,m}(x) \leq \|H^{m+3/2} x\| + c\,\|H^{m+1} x\| \leq (1+c) \|H^{m+3/2} x\| \quad , \tag{3.8}$$

where $c = 2\,d^2\,M(1+2M)$, for all $m \in \mathbf{N}_0$ together with the recursive inequalities

$$M_{n,m}(x) \leq b\,M_{n-2,m+1}(x) + c\,n\,M_{n-1,m}(x) \quad , \tag{3.9}$$

where $b = 1 + 2M$, for all $n, m \in \mathbf{N}_0$ with $n \geq 2$ and all $x \in \mathcal{H}_\infty$. The remainder of the proof relies on 'solving' (3.8) and (3.9) for $x \in \mathcal{H}_\omega(H^{1/2}) \subseteq \mathcal{H}_\infty$.

First, if $x \in \mathcal{H}_\omega(H^{1/2})$ there are $a_0 > 0$ and $t_0 \geq 1$ such that

$$\|H^{m/2} x\| \leq a_0\, t_0^m\, m! \tag{3.10}$$

for all $m \in \mathbf{N}_0$. Therefore, one has

$$M_{0,m}(x) = \|H^{m+1} x\| \leq a_0\, t_0^{2m+2}\, (2m+2)! \leq (3 a_0 t_0^2)\, (2t_0)^{2m}\, (2m)!$$

for all $m \in \mathbf{N}_0$ where we have used (3.10). Moreover, (3.8) gives

$$M_{1,m}(x) \leq (1+c) \|H^{m+3/2} x\| \leq (3(1+c) a_0 t_0^2)\, (2t_0)^{2m+1}\, (2m+1)!$$

for all $m \in \mathbf{N}_0$ where we have again used (3.10). Therefore

$$M_{n,m}(x) \leq a_1\, s^n\, t_1^{2m}\, (2m+n)!$$

for $n \in \{0, 1\}$, $m \in \mathbf{N}_0$ and all $s \geq 1$ with $a_1 = 3a_0(1+c)t_0^2$ and $t_1 = 2t_0$.

Secondly, let $N \geq 2$ and suppose

$$M_{n,m}(x) \leq a_1\, s^n\, t_1^{2m}\, (2m+n)!$$

for all $n < N$, $m \in \mathbf{N}_0$ and $s \geq 1$. Then (3.9) gives

$$\begin{aligned} M_{N,m}(x) &\leq a_1\, s^N\, t_1^{2m}\, (2m+N)! \Big(b\, s^{-2}\, t_1^2 + c\, s^{-1} N (2m+N)^{-1} \Big) \\ &\leq a_1\, s^N\, t_1^{2m}\, (2m+N)!\, (b\, t_1^2 + c)\, s^{-1} \end{aligned}$$

for all $s \geq 1$. But $b\,t_1^2 + c \geq b \geq 1$. Hence if $s = b\,t_1^2 + c$ then

$$M_{n,m}(x) \leq a_1\, s^n\, t_1^{2m}\,(2m+n)! \tag{3.11}$$

for $n = N$ and all $m \in \mathbf{N}_0$. Therefore one concludes by inductive reasoning that (3.11) is valid for all $m, n \in \mathbf{N}_0$.

Thirdly, specializing (3.11) to the case $m = 0$, one deduces that

$$\|A^\alpha x\| \leq \|HA^\alpha x\| = M_{|\alpha|,0}(x) \leq a_1\, s^{|\alpha|}\, |\alpha|!$$

for all α. Hence $x \in \mathcal{H}_\omega(U)$ by Lemma 2.4 and $\mathcal{H}_\omega(H^{1/2}) \subseteq \mathcal{H}_\omega(U)$.

Finally it follows from spectral theory that the space of analytic elements for any self-adjoint operator is dense. Hence the density of the subspace $\mathcal{H}_\omega(U)$ follows from the density of $\mathcal{H}_\omega(H^{1/2})$. □

The advantage of the foregoing discussion is that the main conclusions are based on general features of the representation which are largely independent of the Hilbert space setting. Hence many of the arguments and conclusions extend to Banach space representations of the group.

4 Banach space representations

The structure of general Banach space representations of a Lie group G is very similar to the structure of the unitary representations discussed in the previous section. There are, however, some significant differences. The C^∞-, and analytic, elements are again characterized by the Laplacian but the C^n-subspaces do not always coincide with the domains of powers of the Laplacian. The difficulty is not an algebraic problem but an analytic one. The usual Laplacian $\Delta = -\sum_{i=1}^d \partial^2/\partial x_i^2$ on $L_2(\mathbf{R}^d\,; dx)$ is closed and its domain coincides with the twice L_2-differentiable functions, in accord with Corollaries 3.2 and 3.4 applied to the unitary representation of $G = \mathbf{R}^d$ acting by translations on L_2. But the Laplacian on $L_1(\mathbf{R}^d\,; dx)$ is not closed and the domain of its closure contains some functions which are not twice differentiable in the L_1-sense [13]. Nevertheless all functions in the domain of the Laplacian on L_1 are once L_1-differentiable and the derivatives are L_1-Hölder continuous with Hölder exponent arbitrarily close to one, i.e., the domain consists of functions which are 'almost' twice differentiable. A similar situation occurs with Banach space representations of a general Lie group. There can be a slight mismatch between the domain of the closure of the Laplacian and the C^2-subspace. A similar mismatch then occurs for the domains of higher powers of the Laplacian. But this small discrepancy is no longer evident at the level of the C^∞-, or analytic, elements. The latter elements are again characterized by the Laplacian in the same manner already seen for unitary representations. But the proofs have to take into account the differences in the differential structures.

Let $(\mathcal{X}, G, U)$ denote a continuous Banach space representation of the Lie group G and $A_i = dU(a_i)$ the representatives of the basis $a_1, \ldots, a_d$ of the Lie algebra $\mathfrak{g}$. The Laplacian Δ is again defined as $\Delta = -\sum_{i=1}^d A_i^2$ with domain $D(\Delta) = \bigcap_{i=1}^d D(A_i^2)$. Therefore $\mathcal{X}_\infty \subseteq D(\Delta)$ and the Laplacian is densely defined. But the adjoint is also densely defined since its domain contains the C^∞-subspace of the adjoint representation. Consequently Δ is closable but it is not generally closed [13]. In a unitary representation Δ is a positive self-adjoint operator, Corollary 3.4, and hence generates a continuous semigroup which is holomorphic in the open right half-plane. These latter properties are a general characteristic and give some basic dissipativity estimates which replace the positivity.

Theorem 4.1 *The Laplacian Δ is closable and its closure $\overline{\Delta}$ generates a continuous semigroup S which is holomorphic in the open right half-plane.*

Moreover, there are $m, \lambda_0 > 0$ such that

$$\|(\lambda I + \overline{\Delta})x\| \geq m\|x\| \tag{4.1}$$

for all $\lambda \geq \lambda_0$ and all $x \in D(\overline{\Delta})$.

The generation result, which is the basis of all the subsequent analysis, can be proved in several ways [11], [12], [14]. A short proof along the lines of the proof of Theorem 3.3 is given in [7]. The principal idea behind the proof of [11], and its variants in [14] and [7], is to approximate G locally by $\mathbf{R}^d$ and then to lift the comparable result for $\mathbf{R}^d$ to G by some form of parametrix argument. This form of reasoning is superficially similar to perturbation theory.

Once one has established that $\overline{\Delta}$ generates a continuous semigroup the bounds (4.1) follow by general semigroup theory. Continuity implies growth bounds $\|S_t\| \leq Me^{\omega t}$ and then Laplace transformation gives the resolvent bounds

$$\|(\lambda I + \overline{\Delta})^{-1}x\| \leq M(\lambda - \omega)^{-1}\|x\|$$

for all $\lambda > \omega$ and $x \in \mathcal{X}$. These readily yield (4.1). The growth bounds also allow one to use standard functional analytic techniques to define fractional powers of $(\lambda I + \overline{\Delta})$ if $\lambda > \omega$ which are useful for the detailed discussion of the analytic structure.

The parametrix arguments used to pass from $\mathbf{R}^d$ to G transform information about the action of the usual Laplacian on $L_2(\mathbf{R}^d\,; dx)$ into information about the Laplacian on $\mathcal{X}$. In particular the arguments yield fairly detailed properties of the domain of the Laplacian. The simplest of these is the 'elliptic regularity' property $D(\overline{\Delta}) \subseteq \mathcal{X}_1$. Thus $\mathcal{X}_2 \subseteq D(\overline{\Delta}) \subseteq \mathcal{X}_1$. These inclusions denote continuous embeddings of Banach spaces expressed by corresponding norm inequalities. In fact the parametrix gives precise quantitative estimates of the form

$$\|x\|_1 \leq \varepsilon\,\|\overline{\Delta}x\| + c\,\varepsilon^{-1}\|x\| \tag{4.2}$$

for all $\varepsilon \in \langle 0,1]$ and $x \in D(\overline{\Delta})$, comparable to the first estimates of Proposition 3.1. There are analogous conclusions for powers of $\overline{\Delta}$. If $n \in \mathbf{N}$ then $\mathcal{X}_{2n} \subseteq D(\overline{\Delta}^n) \subseteq \mathcal{X}_{2n-1}$ and

$$\|x\|_m \le \varepsilon \, \|\overline{\Delta}^n x\| + c_n \, \varepsilon^{-m/(2n-m)} \|x\|$$

for some $c_n > 0$, all $\varepsilon \in \langle 0,1]$, $x \in D(\overline{\Delta}^n)$ and $m \in \{1,2,\ldots,2n-1\}$. Even more detailed properties can be deduced but only these simplest aspects of elliptic regularity are sufficient to elucidate the C^∞-structure.

Theorem 4.2 *The C^∞-elements of U coincide with those of $\overline{\Delta}$,*

$$\mathcal{X}_\infty(U) = \bigcap_{n=1}^{\infty} D(\overline{\Delta}^n) = D_\infty(\overline{\Delta}) \quad .$$

Hence $\mathcal{X}_\infty(U)$ is a core of Δ.

Proof The identification of the two sets of C^∞-elements follows from the inclusions

$$\mathcal{X}_{2n} \subseteq D(\overline{\Delta}^n) \subseteq \mathcal{X}_{2n-1}$$

by taking intersections over n. As the semigroup S is holomorphic it follows that S_t maps $D_\infty(\overline{\Delta})$ into itself for all $t > 0$ and hence $D_\infty(\overline{\Delta}) = \mathcal{X}_\infty(U)$ is a core of Δ. □

The characterization of the analytic elements by the Laplacian is more difficult and requires better understanding of the C^n-structure. The earlier discussion of unitary representations was based on an identification $D(\overline{\Delta}) = \mathcal{X}_2$ but this is only valid for special representations. We will discuss a second important class below, the Lipschitz representations. The identification $D(\overline{\Delta}) = \mathcal{X}_2$ is equivalent to $\Delta|_{\mathcal{X}_2}$ being closed and this property fails for the left regular representation of G on $L_1(G\,; dg)$. If, however, $D((\lambda I + \overline{\Delta})^{1/2}) = \mathcal{X}_1$ for large λ then one has $D((\lambda I + \overline{\Delta})^{n/2}) = \mathcal{X}_n$ for all $n \in \mathbf{N}$ (see [6] Corollary 3.14). But the validity of the identity for $n = 2$ does not necessarily imply its validity for $n = 1$. In the left regular representation of $\mathbf{R}$ in $L_1(\mathbf{R})$ the Laplacian Δ is closed, but $D((\lambda I + \overline{\Delta})^{1/2}) = \mathcal{X}_1$ fails since the Riesz transform is not bounded on $L_1(\mathbf{R})$.

The key to understanding the details of the differential structure lies in the Lipschitz substructure. Elements in the domain of $\overline{\Delta}$ are once differentiable, $D(\overline{\Delta}) \subseteq \mathcal{X}_1$, but in addition the derivatives are Hölder continuous in the sense that

$$\sup_{0<|g|\le 1} |g|^{-\gamma} \|(I - U(g)) A_i x\| < \infty$$

for each $i \in \{1,\ldots,d\}$ and $\gamma \in \langle 0,1\rangle$ where $|\cdot|$ is some modulus on the group G. A more precise statement can be made in terms of the Lipschitz spaces $\mathcal{X}_{n+\gamma}$, where $n \in \mathbf{N}$ and $\gamma \in \langle 0,1\rangle$, defined as the subspaces of $\mathcal{X}$ for which the corresponding norm

$$\|x\|_{n+\gamma} = \|x\| + \sup_{0<|g|\le 1} \max_{\alpha;\, |\alpha|\le n} |g|^{-\gamma} \|(I - U(g)) A^\alpha x\| \tag{4.3}$$

is finite. Since

$$\|(I - U(g))A^{\alpha}x\| \leq c\,|g| \cdot \|x\|_{n+1}$$

for suitable $c > 0$ it follows that the Lipschitz spaces are intermediate to the C^n-spaces, i.e.,

$$\mathcal{X}_{n+1} \subseteq \mathcal{X}_{n+\gamma} \subseteq \mathcal{X}_n \quad .$$

Moreover, the Lipschitz construction is transitive with respect to the family of C^n-subspaces, e.g., the $(n+\gamma)$-Lipschitz space formed with respect to the representation $U^{(m)}$ of G on $\mathcal{X}_m$ is the space $\mathcal{X}_{m+n+\gamma}$. In fact $\|\cdot\|_{n+\gamma}$ is equivalent to the norm

$$x \mapsto \|x\|'_{n+\gamma} = \|x\|_n + \sup_{0<|g|\leq 1} |g|^{-\gamma}\|(I - U(g))x\|_n \quad .$$

These various properties clearly indicate that the index $n+\gamma$ corresponds to a fractional order of differentiability.

The Lipschitz spaces give the possibility of delineating more detailed domain properties of the Laplacian. One can establish that

$$\mathcal{X}_{2n} \subseteq D(\overline{\Delta}^n) \subseteq \mathcal{X}_{2n-1+\gamma} \quad .$$

This is one way of expressing the fact that $D(\overline{\Delta}^n)$ is very nearly equal to $\mathcal{X}_{2n}$, that elements of $D(\overline{\Delta}^n)$ are very close to being $2n$-times differentiable.

The Lipschitz spaces also give a different method of describing regularity. Each space $\mathcal{X}_{n+\gamma}$ is automatically invariant under the representation U. In particular one has a family of representations $U_{n+\gamma} = U|_{\mathcal{X}_{n+\gamma}}$. But these representations are not usually continuous because of the use of the supremum in the definition of the norm (4.3). Hence it is convenient to modify this definition. If $q \in [1,\infty\rangle$ then the spaces $\mathcal{X}_{n+\gamma,q}$ are defined as the subspaces of $\mathcal{X}$ for which the corresponding norm

$$\|x\|_{n+\gamma,q} = \|x\| + \max_{\alpha;\,|\alpha|\leq n} \left(\int_{\Omega_e} dg\,|g|^{-d}\Big(|g|^{-\gamma}\|(I - U(g))A^{\alpha}x\|\Big)^q \right)^{1/q}$$

is finite where Ω_e is a bounded open neighbourhood of the identity. These spaces are again intermediate to the C^n-subspaces and have similar transitivity properties relative to the C^n-subspaces as those described above. In addition one again has precise embedding properties for the Laplacian

$$\mathcal{X}_{2n} \subseteq D(\overline{\Delta}^n) \subseteq \mathcal{X}_{2n-1+\gamma,q} \quad .$$

The modified Lipschitz spaces are U-invariant but now the representations $U_{\gamma,q} = U|_{\mathcal{X}_{\gamma,q}}$ are continuous. The interesting feature of these representations is that they are, in general, more regular than the original representation or the representations associated with the C^n-subspaces.

Consider the spaces $\mathcal{X}_{\gamma,q}$. These spaces are intermediate to $\mathcal{X}$ and $\mathcal{X}_1$,

$$\mathcal{X}_1 \subseteq \mathcal{X}_{\gamma,q} \subseteq \mathcal{X} \quad ,$$

and the embeddings are continuous; there are $c_{\gamma,q} > 0$ such that

$$\|x\| \leq \|x\|_{\gamma,q} \leq c_{\gamma,q}\|x\|_1$$

for all $x \in \mathcal{X}_1$. These latter inequalities are essential in what follows. Now one can associate with the spaces $\mathcal{X}_{\gamma,q}$ and the corresponding representations $U_{\gamma,q}$ families of C^n-subspaces $\mathcal{X}_{\gamma,q;n}$ etc.. Let $\Delta_{\gamma,q}$ denote the closed Laplacian associated with the representation $U_{\gamma,q}$, and the basis $a_1, \ldots, a_d$. Since $U_{\gamma,q}$ is obtained by restriction of U the Lipschitz Laplacian $\Delta_{\gamma,q}$ is obtained by restriction of $\overline{\Delta}$. But it is convenient to retain the notational distinction. The operator $\Delta_{\gamma,q}$ generates a continuous semigroup on $\mathcal{X}_{\gamma,q}$ and satisfies dissipativity bounds

$$\|(\lambda I + \Delta_{\gamma,q})x\|_{\gamma,q} \geq m\|x\|_{\gamma,q} \tag{4.4}$$

for all large λ analogous to (4.1). The important point, however, is that the Lipschitz Laplacian also satisfies the regularity bounds

$$\|x\|_{\gamma,q;1} \leq a_{\gamma,q}\|(\lambda I + \Delta_{\gamma,q})^{1/2}x\|_{\gamma,q} \tag{4.5}$$

and

$$\|x\|_{\gamma,q;2} \leq b_{\gamma,q}\|(\lambda I + \Delta_{\gamma,q})x\|_{\gamma,q} \quad , \tag{4.6}$$

for λ sufficiently large, analogous to the bounds coming from Corollary 3.5. This is the surprising element of the Lipschitz representations, their increased regularity. The regularity of the Lipschitz spaces is well known in the by now classical theory of function spaces over $\mathbf{R}^d$ (see, for example, [15], Chapter V, or [16]) but it is only recently that the importance of these properties for the Lie group theory has been emphasized [14].

The regularity properties (4.5) and (4.6) allow one to deduce that the domains of the powers of the Lipschitz Laplacians $\Delta_{\gamma,q}$ coincide with the C^n-subspaces of the Lipschitz spaces,

$$D(\Delta^n_{\gamma,q}) = \mathcal{X}_{\gamma,q;2n} \quad ,$$

by the argument used to prove Theorem 3.6. Moreover, the estimates (4.4), (4.5) and (4.6) suffice to prove the Lipschitz equivalent of Theorem 3.9.

Theorem 4.3 *If $(\mathcal{X}_{\gamma,q}, G, U_{\gamma,q})$ is a Lipschitz representation associated with the Banach space representation $(\mathcal{X}, G, U)$ then*

$$\mathcal{Y}_\omega(U_{\gamma,q}) = \mathcal{Y}_\omega((\lambda I + \Delta_{\gamma,q})^{1/2})$$

for all large λ, where $\mathcal{Y} = \mathcal{X}_{\gamma,q}$.

It might appear that Theorem 4.3 misses the point since it does not give any direct statement about the analytic structure of the underlying Banach space representation $(\mathcal{X}, G, U)$. But it immediately leads to the desired conclusion by a straightforward embedding argument.

Corollary 4.4 *If $(\mathcal{X}, G, U)$ is a continuous Banach space representation then*

$$\mathcal{X}_\omega(U) = \mathcal{X}_\omega((\lambda I + \overline{\Delta})^{1/2})$$

for all large λ and the subspace $\mathcal{X}_\omega(U)$ of analytic elements is dense in $\mathcal{X}$.

Proof If $x \in \mathcal{X}_\omega((\lambda I + \overline{\Delta})^{1/2})$ there are $a, t > 0$ such that

$$\|(\lambda I + \overline{\Delta})^{n/2} x\| \le a\, t^n n!$$

for all $n \in \mathbf{N}_0$. But

$$\|(\lambda I + \Delta_{\gamma,q})^{n/2} x\|_{\gamma,q} \le c\, \|(\lambda I + \overline{\Delta})^{n/2} x\|_1$$

because $\mathcal{X}_{\gamma,q}$ is continuously embedded in $\mathcal{X}_1$ and $U_{\gamma,q}$ is the restriction of U. The embedding $D(\overline{\Delta}) \subseteq \mathcal{X}_1$ gives, however, continuity estimates

$$\|y\|_1 \le c'\, \|(\lambda I + \overline{\Delta}) y\|$$

for all $y \in D(\overline{\Delta})$, e.g., these follow by combination of (4.1) and (4.2). Therefore

$$\|(\lambda I + \Delta_{\gamma,q})^{n/2} x\|_{\gamma,q} \le c''\, \|(\lambda I + \overline{\Delta})^{n/2+1} x\| \le a c''\, t^{n+2}\, (n+2)! \le a_1\, t_1^n\, n!$$

with $a_1 = 3ac''t^2$ and $t_1 = 2t$. Thus $x \in \mathcal{Y}_\omega((\lambda I + \Delta_{\gamma,q})^{1/2}) = \mathcal{Y}_\omega(U_{\gamma,q})$ and there are $b, s > 0$ such that

$$\|x\|_{\gamma,q;n} \le b\, s^n\, n!$$

for all $n \in \mathbf{N}_0$. Since $\|x\|_n \le \|x\|_{\gamma,q;n}$ it follows that $x \in \mathcal{X}_\omega(U)$ and this establishes that $\mathcal{X}_\omega((\lambda I + \overline{\Delta})^{1/2}) \subseteq \mathcal{X}_\omega(U)$.

The converse inclusion follows by a similar argument. But it can also be established by a simple direct argument based on the bounds $\|\overline{\Delta}^n x\| \le d^n\, \|x\|_{2n}$.

Finally, since $(\lambda I + \overline{\Delta})^{1/2}$ generates a holomorphic semigroup the space $\mathcal{X}_\omega(U)$ is dense in $\mathcal{X}$. □

In conclusion Theorem 4.2 and Corollary 4.4 establish that the C^∞-, and analytic, structures of a general Banach space representation are characterized by the Laplacian in the same manner found earlier for unitary representations.

The characterization of the C^n-structure by the Laplacian established in Theorem 3.6 for unitary representations is also valid for many other representations. It holds for the Lipschitz representations and is also valid for principal series representation of a semi-simple Lie group [5] or for the left regular representation of G in $L_p(G)$ if $1 < p < \infty$ [2]. It is an interesting question whether it is possible to characterize those representations for which the differential structure is determined by the Laplacian in this manner.

Acknowledgements

The second-named author had the good fortune to be a NATO Research Fellow in the Institut für Theoretische Physik at the ETH, Zürich, from 1960 to 1962. Under the leadership of Res Jost and Markus Fierz the Institute, then located in Hochstrasse, provided a stimulating atmosphere in which many diverse topics of mathematics and physics, including analytic elements of Lie groups, were warmly debated by all members including graduate students Klaus Hepp and Walter Hunziker.

Both authors have pleasure in dedicating the current paper to Klaus and Walter as they reach their sixtieth birthdays.

Part of this work was carried out whilst the second-named author was visiting Hokkaido University as a guest of Akitaka Kishimoto. This visit was made possible by the exchange program of the Australian Academy of Science and the Japan Society for the Promotion of Science.

References

[1] Bratteli, O., and Robinson, D.W., *Operator algebras and quantum statistical mechanics*, vol. 1. Second edition. Springer-Verlag, New York etc., 1987.

[2] Burns, R.J., Elst, A.F.M. ter, and Robinson, D.W., L_p-regularity of subelliptic operators on Lie groups. *J. Operator Theory* **31** (1994), 165–187.

[3] Cartier, P., and Dixmier, J., Vecteurs analytiques dans les représentations de groupes de Lie. *Amer. J. Math.* **80** (1958), 131–145.

[4] Dixmier, J., and Malliavin, P., Factorisations de fonctions et de vecteurs indéfiniment différentiables. *Bull. Sc. Math., 2e série* **102** (1978), 305–330.

[5] Elst, A.F.M. ter, On the differential structure of principal series representations. *J. Operator Theory* **28** (1992), 309–320.

[6] Elst, A.F.M. ter, and Robinson, D.W., Subcoercivity and subelliptic operators on Lie groups I: Free nilpotent groups. *Potential Anal.* **3** (1994), 283–337.

[7] ——, Elliptic operators on Lie groups. *Acta Appl. Math.* (1996). To appear.

[8] Gårding, L., Note on continuous representations of Lie groups. *Proc. Nat. Acad. Sc. U.S.A.* **33** (1947), 331–332.

[9] ——, Vecteurs analytiques dans les représentations des groupes de Lie. *Bull. Soc. Math. France* **88** (1960), 73–93.

[10] Kato, T., *Perturbation theory for linear operators.* Second edition, Grundlehren der mathematischen Wissenschaften 132. Springer-Verlag, Berlin etc., 1984.

[11] Langlands, R.P., *Semi-groups and representations of Lie groups.* PhD thesis, Yale University, 1960. (Unpublished).

[12] Nelson, E., Analytic vectors. *Ann. Math.* **70** (1959), 572–615.

[13] Ornstein, D., A non-inequality for differential operators in the L_1 norm. *Arch. Rational Mech. Anal.* **11** (1962), 40–49.

[14] Robinson, D.W., *Elliptic operators and Lie groups.* Oxford Mathematical Monographs. Oxford University Press, Oxford etc., 1991.

[15] Stein, E.M., *Singular integrals and differential properties of functions.* Princeton Mathematical Series 30. Princeton University Press, Princeton, New Jersey, 1970.

[16] Triebel, H., *Interpolation theory, function spaces, differential operators.* North-Holland, Amsterdam, 1978.

[17] Varopoulos, N.T., Saloff-Coste, L., and Coulhon, T., *Analysis and geometry on groups.* Cambridge Tracts in Mathematics 100. Cambridge University Press, Cambridge, 1992.