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Flux and Scattering Into Cones in Potential Scattering

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Abstract. For short range potentials we prove that the probability of finding a scattered quantum-mechanical particle at large times in a truncated cone is identical with the scattered flux, integrated over time, across a distant spherical surface subtending this cone.

1 Introduction

The definition of scattering cross sections in the Hilbert space approach to quantum scattering theory is based on Dollard's scattering into cones formula (see *e.g.* [1, 2, 3]), whereas in standard texts on quantum mechanics the scattering cross section is usually introduced in terms of probability currents across distant surfaces (*e.g.* in [4]). As argued in [5] and [6], the latter approach represents a better reflection of the experimental situation where particles get counted in a detector, and it is interesting to study the relation between the two points of view. The present paper is concerned with this problem.

We consider the simple case of potential scattering in $\mathbb{R}^n$. Let $H_0 = -\Delta$ be the free Hamiltonian and $H = -\Delta + V(\boldsymbol{x})$ the total Hamiltonian, and denote by $\{U_t^0\}_{t\in\mathbb{R}}$ and $\{U_t\}_{t\in\mathbb{R}}$ the associated evolution groups in the Hilbert space $\mathscr{H} = L^2(\mathbb{R}^n)$, *i.e.* $U_t^0 = \exp(-iH_0t)$ and $U_t = \exp(-iHt)$ respectively. For a large class of short range potentials, the wave operators $\Omega_\pm = \text{s-lim}_{t\to\pm\infty} U_t^* U_t^0$ exist and the scattering operator $S = \Omega_+^*\Omega_-$ is unitary. If $f \in \mathscr{H}$ is interpreted as an initial state vector, then Sf is the associated final state

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vector and $g = \Omega_- f = \Omega_+ Sf$ the associated scattering state vector, in the sense that $\lim_{t\to-\infty} \| U_t\, g - U_t^0 f \| = 0$ and $\lim_{t\to+\infty} \| U_t\, g - U_t^0 Sf \| = 0$, and $g_t(\boldsymbol{x}) = [U_t \Omega_- f](\boldsymbol{x})$ represents the full wave function at time t.

Let $\mathcal{C}$ be a cone in $\mathbb{R}^n$ with vertex at the origin and R a positive real number. We define $\mathcal{C}_R$ to be the truncated cone $\mathcal{C}_R = \{\boldsymbol{x} \in \mathcal{C} \mid r > R\}$ †, and we denote by Σ_R the intersection of the cone $\mathcal{C}$ with the sphere $\mathcal{S}_R = \{\boldsymbol{x} \in \mathbb{R}^n \mid r = R\}$, *i.e.* $\Sigma_R = \{\boldsymbol{x} \in \mathcal{C} \mid r = R\}$. Then the probability that the scattered state be localized in the truncated cone $\mathcal{C}_R$ after the scattering event is given by

$$\mathrm{P}(f, \mathcal{C}_R) = \lim_{t\to+\infty} \int_{\mathcal{C}_R} |g_t(\boldsymbol{x})|^2 \, d^n x \tag{1.1}$$

(if one assumes that $\|f\| = 1$). Dollard's scattering into cones theorem states that this quantity is identical with the probability that the momentum of the final state lies in the cone $\mathcal{C}$, *i.e.* one has

$$\mathrm{P}(f, \mathcal{C}_R) = \int_{\mathcal{C}} |\widetilde{Sf}(\boldsymbol{k})|^2 \, d^n k \tag{1.2}$$

for each $f \in \mathscr{H}$. A further argument, involving an average over a collection of initial state vectors f describing a *beam*, then leads to an expression for the scattering cross section for the cone $\mathcal{C}$ (the details of this are irrelevant in the context of this paper).

The second quantity of interest is the scattered flux across Σ_R at time t, given by

$$\phi_{\Sigma_R}(g, t) = 2\,\mathrm{Im} \int_{\Sigma_R} \overline{g_t(\boldsymbol{x})}\, \boldsymbol{\nabla} g_t(\boldsymbol{x}) \cdot \boldsymbol{n}(\boldsymbol{x}) \, d\sigma \, ; \tag{1.3}$$

here $\boldsymbol{n}(\boldsymbol{x}) = \boldsymbol{x}/r$ is the outward unit vector orthogonal to $\mathcal{S}_R$ at $\boldsymbol{x}$ and $d\sigma$ the surface (Lebesgue) measure on $\mathcal{S}_R$. The sign of ϕ_{Σ_R} indicates whether the flux is outgoing ($\phi_{\Sigma_R} > 0$) or incoming ($\phi_{\Sigma_R} < 0$).

One expects that, if R is sufficiently large, then $\mathrm{P}(f, \mathcal{C}_R)$ should be the same as the total scattered flux (the scattered flux integrated over time) across Σ_R, more precisely that, for any $T \in \mathbb{R}$:

$$\lim_{R\to\infty} \int_T^\infty \phi_{\Sigma_R}(\Omega_- f, t) \, dt = \int_{\mathcal{C}} |\widetilde{Sf}(\boldsymbol{k})|^2 \, d^n k \, . \tag{1.4}$$

The equality of the two quantities appearing in (1.4) was conjectured, for N-body systems, by Combes *et al.* in [6], and recently the validity of this conjecture has been explicitly verified for free particles (*i.e.* with $V = 0$, hence $\Omega_\pm = S = I$, the identity operator) by Daumer *et al.* in [7]. Our purpose here is to prove (1.4) for a large class of short range potentials V, and we plan to consider more complicated situations in a forthcoming publication.

Our method consists in showing that, as $R \to \infty$, the scattered flux across Σ_R converges to the free flux associated to the final state vector Sf. The idea of the proof is explained in Section 2, but the technical estimates are deferred to Section 3; these estimates rely on asymptotic properties ($t \to +\infty$) of the evolution group $U_t = \exp(-iHt)$ which we gather

†We use the notation $r = |\boldsymbol{x}| = \left(\sum_{j=1}^n x_j^2\right)^{1/2}$.

from earlier publications. This proof does not cover the situation of slowly decaying (short range) potentials; in this case it is necessary to use a modified free evolution, as is shown in Section 4. The principal results are stated in Theorems 1 and 2, and in an appendix we discuss the flux associated to the free and the modified free evolution. A different method (based on properties of eigenfunctions) for obtaining the result (1.4) for $V \neq 0$ is indicated in [7] and applied to potentials of class C_0^∞ in [8].

2 Free flux and scattered flux

We now outline our method of proving (1.4). Precise conditions on the potential V and details on the necessary estimates will be given in Section 3. The following conventions will be used: $\boldsymbol{Q} = (Q_1, \ldots, Q_n)$ and $\boldsymbol{P} = (P_1, \ldots, P_n)$ denote the n-component position and momentum operator respectively (Q_j is multiplication by x_j and $P_j = -i\,\partial/\partial x_j$). We set $\langle Q\rangle = (I + \boldsymbol{Q}^2)^{1/2}$, $\langle Q\rangle^\nu = (I + \boldsymbol{Q}^2)^{\nu/2}$ for $\nu \in \mathbb{R}$, $Q = (\sum_{j=1}^n Q_j^2)^{1/2}$ and $P = (\sum_{j=1}^n P_j^2)^{1/2}$. Each of these operators is considered on the domain on which it is self-adjoint.

We denote by Γ_R the following operator from $L^2(\mathbb{R}^n)$ to $L^2(\mathcal{S}^{n-1})$, where $\mathcal{S}^{n-1} \equiv \mathcal{S}_1 = \{\boldsymbol{x} \in \mathbb{R}^n \mid r = 1\}$:

$$[\Gamma_R u](\boldsymbol{\omega}) = [(I + P)^{-1}u](R\boldsymbol{\omega})\,; \tag{2.1}$$

here $\boldsymbol{\omega} \in \mathcal{S}^{n-1}$ and $u \in L^2(\mathbb{R}^n)$. So $\Gamma_R u$ is the restriction of the function $(I + P)^{-1}u$ to the sphere $\mathcal{S}_R$ of radius R. It is known (see the end of Section 3) that Γ_R is a bounded operator and that there is a constant $c_n < \infty$ (depending on n) such that

$$\| \Gamma_R \| \leqslant \frac{c_n}{R^{(n-1)/2}} \qquad \text{for all } R \geqslant 1. \tag{2.2}$$

The operator Γ_R is useful for expressing the flux across Σ_R associated to a state vector u. One has $d\sigma = R^{n-1}\,d\omega$ ($d\omega$ the Lebesgue measure on $\mathcal{S}^{n-1}$) and $\boldsymbol{n}(\boldsymbol{x})\cdot\boldsymbol{\nabla} = i\sum_{j=1}^n \omega_j P_j$, where ω_j denotes the j-th component of the vector $\boldsymbol{\omega} \in \mathcal{S}^{n-1}$. By recalling that $\Sigma_1 = \mathcal{C} \cap \mathcal{S}^{n-1}$, one obtains for any $\varepsilon \in \mathbb{R}$:

$$\begin{aligned}
\Phi_{\Sigma_R}(u) &\equiv 2\,\mathrm{Im} \int_{\Sigma_R} \overline{u(\boldsymbol{x})}\; \boldsymbol{\nabla} u(\boldsymbol{x})\cdot\boldsymbol{n}(\boldsymbol{x})\, d\sigma \\
&= 2\sum_{j=1}^n \mathrm{Re} \int_{\Sigma_1} R^{n-1}\, d\omega\; \overline{u(R\boldsymbol{\omega})}\; \omega_j [P_j u](R\boldsymbol{\omega}) \\
&= \frac{2R^{n-1}}{(1+R^2)^{\varepsilon/2}} \sum_{j=1}^n \mathrm{Re} \int_{\Sigma_1} d\omega\; \overline{[\langle Q\rangle^\varepsilon u](R\boldsymbol{\omega})}\; \omega_j [P_j u](R\boldsymbol{\omega}) \\
&= \frac{2R^{n-1}}{(1+R^2)^{\varepsilon/2}} \sum_{j=1}^n \mathrm{Re} \int_{\Sigma_1} d\omega\; \overline{\big[\Gamma_R (I+P)\langle Q\rangle^\varepsilon u\big](\boldsymbol{\omega})}\; \omega_j \big[\Gamma_R (I+P) P_j u\big](\boldsymbol{\omega})\,.
\end{aligned} \tag{2.3}$$

Alternatively, one may use the expression

$$\Phi_{\Sigma_R}(u) = \frac{2R^{n-1}}{(1+R^2)^{\varepsilon/2}} \sum_{j=1}^n \mathrm{Re} \int_{\Sigma_1} d\omega\; \overline{\big[\Gamma_R (I+P) u\big](\boldsymbol{\omega})}\; \omega_j \big[\Gamma_R (I+P)\langle Q\rangle^\varepsilon P_j u\big](\boldsymbol{\omega})\,. \tag{2.4}$$

As in the Introduction, consider now an initial state vector f and set $g_t = U_t \Omega_- f$, $h = Sf$ and $h_t = U_t^0 h$. Then $\phi^{0}_{\Sigma_R}(h,t) \equiv \Phi_{\Sigma_R}(h_t)$ represents the free flux of the final state across Σ_R at time t, whereas $\phi_{\Sigma_R}(g,t) \equiv \Phi_{\Sigma_R}(g_t)$ is the scattered flux across Σ_R at time t. We shall show that the integrated scattered flux across Σ_R coincides with the integrated free flux of the final state h across Σ_R in the limit $R \to \infty$.

The properties of the free flux were established in [7] where it is shown that, for any $T \in \mathbb{R}$ and a dense set of state vectors h:

$$\lim_{R\to\infty} \int_T^\infty \phi^{0}_{\Sigma_R}(h,t)\,dt = \lim_{R\to\infty} \int_T^\infty \left|\phi^{0}_{\Sigma_R}(h,t)\right| dt = \int_C |\tilde{h}(\boldsymbol{k})|^2\, d^n k\,. \tag{2.5}$$

To estimate the difference between the scattered and the free flux, we observe that

$$\begin{aligned}
&\phi_{\Sigma_R}(g,t) - \phi^{0}_{\Sigma_R}(h,t) = \\
&\quad \frac{2R^{n-1}}{(1+R^2)^{\varepsilon/2}} \sum_{j=1}^{n} \mathrm{Re} \int_{\Sigma_1} d\omega\, \overline{\left[\Gamma_R(I+P)(g_t-h_t)\right](\boldsymbol{\omega})}\, \omega_j \left[\Gamma_R(I+P)\langle Q\rangle^\varepsilon P_j h_t\right](\boldsymbol{\omega}) \\
&\quad + \frac{2R^{n-1}}{(1+R^2)^{\varepsilon/2}} \sum_{j=1}^{n} \mathrm{Re} \int_{\Sigma_1} d\omega\, \overline{\left[\Gamma_R(I+P)\langle Q\rangle^\varepsilon g_t\right](\boldsymbol{\omega})}\, \omega_j \left[\Gamma_R(I+P)P_j(g_t-h_t)\right](\boldsymbol{\omega}).
\end{aligned}$$

So, by using first the Cauchy-Schwarz inequality and then the estimate (2.2), one obtains that

$$\begin{aligned}
&\left|\phi_{\Sigma_R}(\Omega_- f,t) - \phi^{0}_{\Sigma_R}(Sf,t)\right| \equiv \left|\phi_{\Sigma_R}(g,t) - \phi^{0}_{\Sigma_R}(h,t)\right| \\
&\quad \leqslant \frac{2R^{n-1}}{(1+R^2)^{\varepsilon/2}} \sum_{j=1}^{n} \Big[\left\|\Gamma_R(I+P)(g_t-h_t)\right\|_{L^2(\Sigma_1)} \left\|\Gamma_R(I+P)\langle Q\rangle^\varepsilon P_j h_t\right\|_{L^2(\Sigma_1)} \\
&\qquad\qquad + \left\|\Gamma_R(I+P)\langle Q\rangle^\varepsilon g_t\right\|_{L^2(\Sigma_1)} \left\|\Gamma_R(I+P)P_j(g_t-h_t)\right\|_{L^2(\Sigma_1)} \Big] \\
&\quad \leqslant \frac{2c_n^2}{(1+R^2)^{\varepsilon/2}} \sum_{j=1}^{n} \Big[\left\|(I+P)(g_t-h_t)\right\| \left\|(I+P)\langle Q\rangle^\varepsilon P_j h_t\right\| \\
&\qquad\qquad + \left\|(I+P)\langle Q\rangle^\varepsilon g_t\right\| \left\|(I+P)P_j(g_t-h_t)\right\| \Big]\,.
\end{aligned} \tag{2.6}$$

If the potential decays faster than r^{-2}, one can exhibit a dense set $\mathscr{M}$ of vectors f for which the norms on the R. H. S. of (2.6) can be majorized as follows:

1. if $f \in \mathscr{M}$, there is a constant c (depending on f) such that for all $t \in \mathbb{R}$ and $\varepsilon \in [0,1]$:

$$\left\|(I+P)\langle Q\rangle^\varepsilon P_j h_t\right\| + \left\|(I+P)\langle Q\rangle^\varepsilon g_t\right\| \leqslant c\,(1+|t|)^{3\varepsilon}\,, \tag{2.7}$$

2. if $f \in \mathscr{M}$ and $T \in \mathbb{R}$, there is a constant c_T such that for all $t \geqslant T$ and some $\delta > 0$:

$$\left\|(I+P)(g_t-h_t)\right\| + \left\|(I+P)P_j(g_t-h_t)\right\| \leqslant c_T(1+|t|)^{-1-\delta}\,. \tag{2.8}$$

The proof of (2.7) and (2.8) will be based on propagation estimates obtained by commutator methods in an earlier paper. By choosing $\varepsilon < \frac{\delta}{3}$, one deduces from (2.6)–(2.8) that, for $f \in \mathscr{M}$:

$$\lim_{R\to\infty} \int_T^\infty \left|\phi_{\Sigma_R}(\Omega_- f, t) - \phi^0_{\Sigma_R}(Sf, t)\right| dt = 0\,. \tag{2.9}$$

Combined with (2.5), this equation implies that

$$\lim_{R\to\infty} \int_T^\infty \phi_{\Sigma_R}(\Omega_- f, t)\, dt = \lim_{R\to\infty} \int_T^\infty \left|\phi_{\Sigma_R}(\Omega_- f, t)\right| dt = \int_{\mathcal{C}} |\widetilde{Sf}(\boldsymbol{k})|^2\, d^n k, \tag{2.10}$$

which proves the conjecture (1.4). The first equation in (2.10) also shows that, if R is sufficiently large, the scattered flux is essentially outgoing.

Remark: The identity (2.10) holds for each finite T. Physically T plays the role of the time at which the detection of scattered states is initiated. If the initial state vector has no *momentum support in* $-\mathcal{C}$ (*i.e.* if $\tilde f(\boldsymbol{k}) = 0$ for all $\boldsymbol{k} \in \mathbb{R}^n$ such that $-\boldsymbol{k} \in \mathcal{C}$), then (2.10) is true also for $T = -\infty$. Indeed, the same arguments that lead to (2.10) also allow one to show that, for any $T \in \mathbb{R}$:

$$\lim_{R\to\infty} \int_{-\infty}^T \phi_{\Sigma_R}(\Omega_- f, t)\, dt = -\lim_{R\to\infty} \int_{-\infty}^T \left|\phi_{\Sigma_R}(\Omega_- f, t)\right| dt = -\int_{\mathcal{C}} |\tilde f(-\boldsymbol{k})|^2\, d^n k, \tag{2.11}$$

and the last integral is zero if f has no momentum support in $-\mathcal{C}$.

3 Estimates on the scattered flux

We first prove (2.7) and (2.8) for a class of short range potentials and for a dense set $\mathscr{D}$ of vectors h (then $\mathscr{M}$ will be given as $\mathscr{M} = S^*\mathscr{D}$). We shall use various results from [9]. From now on we assume the following condition to be satisfied:

(H₁) *V is a real-valued function on $\mathbb{R}^n$ of the form*

$$V(\boldsymbol{x}) = (1 + \boldsymbol{x}^2)^{-a/2}\left[W_1(\boldsymbol{x}) + W_2(\boldsymbol{x})\right] \tag{3.1}$$

with $a > 1$, $W_1 \in L^\infty(\mathbb{R}^n)$, $\boldsymbol{x}\cdot \operatorname{grad} W_1 \in L^\infty(\mathbb{R}^n) + L^{q_1}(\mathbb{R}^n)$ and $(1+r)W_2 \in L^\infty(\mathbb{R}^n) + L^{q_2}(\mathbb{R}^n)$, where q_1 and q_2 satisfy $q_j \geqslant 2$ and $q_j > \frac{n}{2}$.

Under this assumption, the Hamiltonian $H = H_0 + V(\boldsymbol{Q})$ is self-adjoint on the domain $D(H_0)$ of H_0.

For $\rho > 0$ we define $\mathscr{D}_\rho$ to be the following dense subset of $\mathscr{H}$:

$$\mathscr{D}_\rho = \left\{h \in L^2(\mathbb{R}^n) \,\middle|\, h \in D(Q^\rho), h = \psi(H_0)h \text{ for some } \psi \in C_0^\infty((0,\infty)\backslash\sigma_p(H))\right\}.$$

We observe that, if $h \in \mathscr{D}_\rho$, then $P_j h \in \mathscr{D}_\rho$, for $j = 1, \ldots, n$.

The following facts have been established in Lemmas 3, 5, 6, 9a and Proposition 4 of [9] ($\mathscr{B}(\mathscr{H})$ denotes the set of all bounded everywhere defined linear operators in $\mathscr{H}$):

Proposition 1:

(a) *For each* $\mu \geqslant 0$, *each* $j = 1, \dots, n$ *and each* $\varphi \in C_0^\infty(\mathbb{R})$, *the operators* $\langle Q\rangle^\mu \varphi(H)\langle Q\rangle^{-\mu}$ *and* $\langle Q\rangle^\mu P_j(H+i)^{-1}\langle Q\rangle^{-\mu}$ *belong to* $\mathscr{B}(\mathscr{H})$.

(b) *For each* $\varphi \in C_0^\infty(\mathbb{R})$, *the closure of the operator* $[\varphi(H) - \varphi(H_0)]\langle Q\rangle^a$ *belongs to* $\mathscr{B}(\mathscr{H})$.

(c) *For each* $\varphi \in C_0^\infty(\mathbb{R})$, *the closure of the operator* $V\varphi(H)\langle Q\rangle^a$ *belongs to* $\mathscr{B}(\mathscr{H})$.

(d) *For each* $\varphi \in C_0^\infty((0,\infty)\backslash\sigma_p(H))$, *each* $\kappa \in [0,a]$ *and each* $\eta > 0$, *there is a constant* c *such that for all* $t \in \mathbb{R}$:

$$\left\|\langle Q\rangle^{-\kappa} U_t \varphi(H)\langle Q\rangle^{-\kappa}\right\| \leqslant c(1+|t|)^{-\kappa+\eta}. \tag{3.2}$$

(e) *Let* $h \in \mathscr{D}_\rho$ *for some* $\rho > 2$ *and assume that* $a > 2$ *in* $(\mathbf{H_1})$. *Then* $\Omega_\pm h \in D(Q^2)$.

Corollary 1: *Assume that* $a > 2$ *in* $(\mathbf{H_1})$ *and let* $\varphi \in C_0^\infty((0,\infty)\backslash\sigma_p(H))$. *Then, for each* $\varepsilon \in [0,2]$ *and each* $\varepsilon' > \varepsilon$, *one has* $\langle Q\rangle^\varepsilon \Omega_\pm \varphi(H_0)\langle Q\rangle^{-\varepsilon'} \in \mathscr{B}(\mathscr{H})$.

Proof. Let $f \in \mathscr{H}$ and $\rho > 2$. Then $\varphi(H_0)\langle Q\rangle^{-\rho} f \in \mathscr{D}_\rho$. Hence $(I+Q^2)\Omega_\pm\varphi(H_0)\langle Q\rangle^{-\rho} f \in \mathscr{H}$. By the closed graph theorem, one has $(I+Q^2)\Omega_\pm\varphi(H_0)\langle Q\rangle^{-\rho} \in \mathscr{B}(\mathscr{H})$, and the assertion of the corollary follows by interpolation. ■

Corollary 2: *Let* κ, $\eta > 0$ *and* $h \in \mathscr{D}_\rho$ *for some* $\rho > 0$. *Then there is a constant* c *such that for each* $t \in \mathbb{R}$:

$$\left\|\langle Q\rangle^{-\kappa} U_t^0 h\right\| \leqslant c(1+|t|)^{-\min(\kappa,\rho)+\eta}. \tag{3.3}$$

Proof. Choose $\psi \in C_0^\infty((0,\infty))$ such that $\psi(H_0)h = h$. Then

$$\left\|\langle Q\rangle^{-\kappa} U_t^0 h\right\| \leqslant \left\|\langle Q\rangle^{-\kappa} U_t^0 \psi(H_0)\langle Q\rangle^{-\rho}\right\| \left\|\langle Q\rangle^\rho h\right\|,$$

and the result follows by using (3.2) for $V = 0$. ■

Proposition 2: *If* $a > 2$, *then for each* $h \in \mathscr{D}_\rho$ *with* $\rho > 2$ *there is a constant* c *such that (2.7) is satisfied.*

Proof. Let $h \in \mathscr{D}_\rho$, $\rho > 2$ and let $\psi \in C_0^\infty((0,\infty)\backslash\sigma_p(H))$ be such that $\psi(H_0)h = h$.

(i) By using the relations

$$[Q_j, U_t^0] = 2tP_jU_t^0 \qquad \text{and} \qquad [Q_j, \psi(H_0)] = 2iP_j\psi'(H_0),$$

and by commuting the powers of $Q_1, \dots, Q_n$ through $U_t^0\psi(H_0)$, one finds that for each $m \in \mathbb{N}$ there is a constant $c(m)$ such that for all $t \in \mathbb{R}$:

$$\left\|(I+Q^2)^m U_t^0\psi(H_0)\langle Q\rangle^{-2m}\right\| \leqslant c(m)(1+|t|)^{2m}.$$

By interpolation this implies that for each $\mu \geqslant 0$ there is a constant $c(\mu)$ such that for all $t \in \mathbb{R}$:

$$\left\|\langle Q\rangle^{\mu} U_t^0 \psi(H_0)\langle Q\rangle^{-\mu}\right\| \leqslant c(\mu)(1+|t|)^{\mu}. \tag{3.4}$$

(ii) Let $\varepsilon \in [0,1]$ be fixed. One has

$$\left\|\langle Q\rangle^{\varepsilon} P_j h_t\right\| \leqslant \left\|\langle Q\rangle^{\varepsilon} U_t^0 \psi(H_0)\langle Q\rangle^{-\varepsilon}\right\|\left\|\langle Q\rangle^{\varepsilon} P_j h\right\| \leqslant c_1(1+|t|)^{\varepsilon},$$

since $P_j h \in \mathscr{D}_\rho \subseteq D(\langle Q\rangle^{\varepsilon})$. Furthermore

$$\begin{aligned} \left\|P_k\langle Q\rangle^{\varepsilon} P_j h_t\right\| &= \left\|\langle Q\rangle^{\varepsilon} P_k P_j h_t - i\varepsilon Q_k\langle Q\rangle^{\varepsilon-2} P_j h_t\right\| \\ &\leqslant \left\|\langle Q\rangle^{\varepsilon} U_t^0 \psi(H_0)\langle Q\rangle^{-\varepsilon}\right\|\left\|\langle Q\rangle^{\varepsilon} P_k P_j h\right\| + \varepsilon\|P_j h\| \\ &\leqslant c_2(1+|t|)^{\varepsilon}. \end{aligned}$$

Since $\|Pf\|^2 = \sum_{k=1}^n \|P_k f\|^2$, the preceding estimates imply that

$$\left\|(I+P)\langle Q\rangle^{\varepsilon} P_j h_t\right\| \leqslant c_3(1+|t|)^{\varepsilon} \qquad \forall\, t \in \mathbb{R}. \tag{3.5}$$

(iii) Similarly one has

$$\left\|(I+P)\langle Q\rangle^{\varepsilon} g_t\right\| \leqslant \left\|\langle Q\rangle^{\varepsilon} g_t\right\| + \sum_{k=1}^{n}\left\|\langle Q\rangle^{\varepsilon} P_k g_t\right\| + \varepsilon\sum_{k=1}^{n}\left\|Q_k\langle Q\rangle^{\varepsilon-2}\right\|\|g_t\|.$$

The last term on the R. H. S. is bounded by a constant independent of t. The first term on the R. H. S. can be estimated by using Corollary 1 and (3.4): if $\varepsilon' \in (\varepsilon, 3\varepsilon) \cap (\varepsilon, \rho)$, then

$$\begin{aligned} \left\|\langle Q\rangle^{\varepsilon} g_t\right\| &\leqslant \left\|\langle Q\rangle^{\varepsilon} \Omega_+ \psi(H_0)\langle Q\rangle^{-\varepsilon'}\right\|\left\|\langle Q\rangle^{\varepsilon'} U_t^0 \psi(H_0)\langle Q\rangle^{-\varepsilon'}\right\|\left\|\langle Q\rangle^{\varepsilon'} h\right\| \\ &\leqslant c_4(1+|t|)^{\varepsilon'} \leqslant c_4(1+|t|)^{3\varepsilon}. \end{aligned}$$

Finally, one has by Proposition 1(a) and the preceding estimate:

$$\begin{aligned} \left\|\langle Q\rangle^{\varepsilon} P_k g_t\right\| &\leqslant \left\|\langle Q\rangle^{\varepsilon} P_k (H+i)^{-1}\langle Q\rangle^{-\varepsilon}\right\|\left\|\langle Q\rangle^{\varepsilon}(H+i)\psi(H)\langle Q\rangle^{-\varepsilon}\right\|\left\|\langle Q\rangle^{\varepsilon} g_t\right\| \\ &\leqslant c_5(1+|t|)^{3\varepsilon}. \end{aligned}$$

■

Proposition 3: *Let $a > 2$, $T \in \mathbb{R}$ and $h \in \mathscr{D}_\rho$ with $\rho > 2$. Let $\delta \in (0,1)$ be such that $\delta < \min(a,\rho) - 2$. Then (2.8) is satisfied for some constant c_T and all $t \geqslant T$.*

Proof. It suffices to show that $\|(I+H_0)(g_t - h_t)\| \leqslant c_T(1+|t|)^{-1-\delta}$ for all $t \geqslant T$. Clearly

$$\|(I+H_0)(g_t - h_t)\| \leqslant \|g\| + \|h\| + \|H_0(H+i)^{-1}\|\|(H+i)g\| + \|H_0 h\| < \infty,$$

so (2.8) is satisfied for all $t \in [T,0]$ (if $T < 0$), and it remains to prove (2.8) for $t > 0$. In this case we write

$$\|g_t - h_t\| = \left\|\psi(H)[U_t\Omega_+ h - U_t^0 h] + [\psi(H) - \psi(H_0)]U_t^0 h\right\|,$$

where $\psi \in C_0^\infty((0,\infty)\backslash\sigma_p(H))$ is such that $\psi(H_0)h = h$. Now by using (3.3) with $\eta = \min(\alpha,\rho) - 2 - \delta$:

$$\begin{aligned}\left\|\psi(H)[U_t\Omega_+h - U_t^0h]\right\| &= \left\|\psi(H)[\Omega_+ - U_t^*U_t^0]h\right\| = \left\|\int_t^\infty U_s^*\psi(H)V\,U_s^0\,h\,ds\right\| \\ &\leqslant \left\|\psi(H)[W_1(\boldsymbol{Q}) + W_2(\boldsymbol{Q})]\right\| \int_t^\infty ds \left\|\langle Q\rangle^{-a}\,U_s^0\,h\right\| \\ &\leqslant c_1\int_t^\infty ds(1+s)^{-\min(a,\rho)+\eta} = c_2(1+t)^{1-\min(a,\rho)+\eta} \\ &= c_2(1+t)^{-1-\delta}.\end{aligned}$$

Furthermore, for any $t\in\mathbb{R}$:

$$\begin{aligned}\left\|[\psi(H)-\psi(H_0)]\,U_t^0h\right\| &\leqslant \left\|[\psi(H)-\psi(H_0)]\langle Q\rangle^a\right\|\left\|\langle Q\rangle^{-a}\,U_t^0\,h\right\| \\ &\leqslant c_3(1+|t|)^{-\min(a,\rho)+\eta},\end{aligned}$$

by Proposition 1(b) and (3.3). Summing up we have for $t>0$:

$$\|g_t - h_t\| \leqslant c_4(1+t)^{-1-\delta}. \tag{3.6}$$

Next define $\varphi\in C_0^\infty((0,\infty)\backslash\sigma_p(H))$ by $\varphi(\lambda) = \lambda\psi(\lambda)$. Then

$$\begin{aligned}\|H_0(g_t-h_t)\| &= \|H_0\psi(H)g_t - H_0\psi(H_0)h_t\| \\ &\leqslant \|H_0\psi(H_0)(g_t-h_t)\| + \|H_0[\psi(H)-\psi(H_0)]g_t\| \\ &\leqslant \|\varphi(H_0)\|\|g_t-h_t\| + \|[\varphi(H)-\varphi(H_0)]g_t\| + \|V\psi(H)g_t\|.\end{aligned} \tag{3.7}$$

The required estimate for the first term on the R. H. S. follows immediately from (3.6). For the second term we use Proposition 1(b,d,e) to get for any $\gamma>0$ and any $t\in\mathbb{R}$:

$$\begin{aligned}\left\|[\psi(H)-\psi(H_0)]g_t\right\| &\leqslant \left\|[\psi(H)-\psi(H_0)]\langle Q\rangle^2\right\|\left\|\langle Q\rangle^{-2}\,U_t\,\psi(H)\langle Q\rangle^{-2}\right\|\left\|\langle Q\rangle^2\,\Omega_+h\right\| \\ &\leqslant c_5(1+|t|)^{-2+\gamma}.\end{aligned}$$

Finally the last term in (3.7) is estimated similarly by using Proposition 1(c,d,e):

$$\begin{aligned}\|V\psi(H)g_t\| &\leqslant \left\|V\psi(H)\langle Q\rangle^a\right\|\left\|\langle Q\rangle^{-a}\,U_t\,\psi(H)\langle Q\rangle^{-2}\right\|\left\|\langle Q\rangle^2\,\Omega_+h\right\| \\ &\leqslant c_6(1+|t|)^{-2+\gamma}.\end{aligned}$$

■

Propositions 2 and 3 show that, if V satisfies $(\mathbf{H_1})$ with $a>2$, then (2.9) is satisfied for each initial state vector f such that $h\equiv Sf$ belongs to $\mathscr{D}_\rho$ for some $\rho>2$. On the other hand, as shown in the Appendix, the proof of (2.5) given in [7] goes through for $h\in\mathscr{D}_\mu$ with $\mu > \frac{n}{2} + [\![\frac{n+1}{2}]\!]$ (*i.e.* $\mu>n$ for n even, and $\mu > n+\frac{1}{2}$ for odd n [‡]). Thus we have:

[‡] If $\nu\in\mathbb{R}$, then $[\![\nu]\!]$ denotes the largest integer $\leqslant \nu$.

Theorem 1: *Assume that V satisfies* $(\mathbf{H_1})$ *with $a > 2$. Then (2.10) holds for each f such that $Sf \in \mathscr{D}_\mu$ for some $\mu > \max(2, \frac{n}{2} + [\![\frac{n+1}{2}]\!])$.*

The preceding result has the inconvenience that the condition imposed on the state vector is in terms of the final state vector $h = Sf$ rather than in terms of the initial state vector. To transcribe this condition into a requirement on the initial state vector f, one has to know suitable mapping properties of the scattering operator S. For a more restricted class of potentials, such properties have recently been obtained in [10]. More precisely: Assume that V has the form (3.1) with $a > 2$ and (i) W_1 of class $C^\infty(\mathbb{R}^n)$ with $|\partial^\alpha W_1(\boldsymbol{x})| \leqslant c_\alpha \langle x\rangle^{-|\alpha|}$ for each multi-index α*, (ii) $\langle x\rangle^{\max(1,n+2-a+\varepsilon)} W_2 \in L^\infty(\mathbb{R}^n) + L^q(\mathbb{R}^n)$ for some $\varepsilon > 0$ and some q satisfying $q \geqslant 2$ and $q > \frac{n}{2}$. Then, if $f \in \mathscr{D}_\rho$, one has $Sf \in \mathscr{D}_\mu$ for $\mu = \min(n+1, \rho)$ (see Theorem 1.4 of [10]). So, under these assumptions, (2.10) holds for each $f \in \mathscr{D}_\rho$ with $\rho > n$ for even n and $\rho > n + \frac{1}{2}$ for odd $n \geqslant 3$.

We end this section with a comment on the bound (2.2) on Γ_R. Its verification is based on the following identity which is rather easy to compute (see *e.g.* Chapter 8 in [11] or § 5.4 in [12]):

$$\Gamma_R^* \Gamma_R = \frac{1}{i\pi R^{n-2}} \operatorname*{w-lim}_{\varepsilon\to+0} (I+P)^{-1}\Big[(\boldsymbol{Q}^2 - R^2 - i\varepsilon)^{-1} - (\boldsymbol{Q}^2 - R^2 + i\varepsilon)^{-1}\Big](I+P)^{-1}.$$

(2.2) is obtained from this identity by using the fact that there is a constant c such that for all $R \geqslant 1$ and all $\varepsilon > 0$:

$$\Big\|(I+P)^{-1}(\boldsymbol{Q}^2 - R^2 \pm i\varepsilon)^{-1}(I+P)^{-1}\Big\| \leqslant \frac{c_n}{R} \tag{3.8}$$

(the proof of (3.8) can be reduced to that of Lemma 5 in § XIII.8 of [13] by interchanging the roles of $\boldsymbol{P}$ and $\boldsymbol{Q}$ and by taking into account the λ-dependence of the constants of the proof in [13]).

4 Short range potentials of slow decay

In the preceding section we had to assume that V satisfies $(\mathbf{H_1})$ with $a > 2$. Under stronger regularity assumptions on V, it is possible to prove (2.10) also for potentials decaying like r^{-a} with $1 < a \leqslant 2$. We now mention some results on this. The assumptions on the potential are as follows:

$(\mathbf{H_2})$ *V is a real-valued function of class C^m on $\mathbb{R}^n$, with $m = \max(n+1, 4)$, and for each multi-index α with $0 \leqslant |\alpha| \leqslant m$ and some $a > 1$:*

$$\Big|[\partial^\alpha V](\boldsymbol{x})\Big| \leqslant c\,(1+r)^{-|\alpha|-a} \qquad \forall\, \boldsymbol{x} \in \mathbb{R}^n, \tag{4.1}$$

where c is a constant.

*If $\alpha = (\alpha_1, \ldots, \alpha_n)$, then $(\partial^\alpha f)(\boldsymbol{z}) = (\frac{\partial}{\partial z_1})^{\alpha_1} \cdots (\frac{\partial}{\partial z_n})^{\alpha_n} f(\boldsymbol{z})$, $\boldsymbol{z}^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n}$ and $|\alpha| = \alpha_1 + \ldots + \alpha_n$.

For $t \in \mathbb{R}$, we define X_t to be the following operator in $\mathscr{B}(\mathscr{H})$:

$$X_t = -\int_t^\infty V(2s\boldsymbol{P})\,ds\,. \tag{4.2}$$

Then one has

$$\Omega_+ = \underset{t\to+\infty}{\text{s-lim}}\, U_t^* U_t^0 = \underset{t\to+\infty}{\text{s-lim}}\, U_t^* U_t^0 e^{-iX_t}\,. \tag{4.3}$$

The arguments of Sections 2 and 3 can be repeated in the present situation with the free evolution U_t^0 replaced by the modified free evolution $U_t^0 e^{-iX_t}$. More precisely, we here let $\overset{0}{\phi}_{\Sigma_R}(h,t)$ be the flux associated to the modified free evolution of h, *i.e.*

$$\overset{0}{\phi}_{\Sigma_R}(h,t) \equiv \Phi_{\Sigma_R}(U_t^0 e^{-iX_t} h)\,.$$

As indicated in the Appendix, this modified free flux still satisfies (2.5) if $\tilde{h}$ is of class C^{n+1} and has compact support in $\mathbb{R}^n\backslash\{0\}$. Furthermore the estimates (2.7) and (2.8) hold for $h_t = U_t^0 e^{-iX_t} h$, $g_t = U_t \Omega_+ h$. Hence we have:

Theorem 2: *Suppose that V satisfies* $(\mathbf{H_2})$. *Then (2.10) holds for each f such that $\widetilde{Sf} \in C_0^m(\mathbb{R}^n\backslash\{0\})$ with $m = \max(n+1, 4)$.*

Again one may use the mapping properties of S given in [10] to replace the condition $\widetilde{Sf} \in C_0^m(\mathbb{R}^n\backslash\{0\})$ by a condition on the initial state vector f. We omit the details but indicate briefly how to prove (2.7) and (2.8) under the assumption that $(\mathbf{H_2})$ holds with $m = 4$ (the condition $m \geqslant n+1$ in $(\mathbf{H_2})$ is needed only in the verification of (2.5)).

The verification of (2.7) in the present situation is done by repeating the arguments of the proof of Proposition 2, with U_t^0 replaced by $U_t^0 e^{-iX_t}$. To obtain the analogue of (3.4) for $\mu \leqslant 4$ (and hence (3.5)), it suffices to observe that the partial derivatives of order $\leqslant 4$ with respect to $\boldsymbol{k}$ of $X_t(\boldsymbol{k}) \equiv -\int_t^\infty V(2s\boldsymbol{k})ds$ are bounded (uniformly in $t \in \mathbb{R}$) on each compact subset of $\mathbb{R}^n\backslash\{0\}$. In part (iii) of the proof one has to know that $\langle Q\rangle^\varepsilon\, \Omega_+ \psi(H_0) \langle Q\rangle^{-\varepsilon'} \in \mathscr{B}(\mathscr{H})$ for some $\varepsilon > 0$ and some $\varepsilon' \leqslant 3\varepsilon$. This is obtained (for $0 < \varepsilon < 1$ and $\varepsilon' > 2\varepsilon$) as in the proof of Corollary 1 from the fact that $\Omega_+\psi(H_0)$ maps $\mathscr{D}_\rho$ into $D(\langle Q\rangle)$ if $\rho > 2$ (see Lemma A.8 in [14]).

The proof of (2.8) in the situation considered here is based on Theorem A.3 of [14]. It is seen from the proof of that theorem that, for each $\eta > 0$ and $h \in \mathscr{D}_\rho$ with $\rho > 3$, there is a constant c_1 such that for all $t \geqslant 0$:

$$\left\| U_t \Omega_+ h - U_t^0 e^{-iX_t} h \right\| \leqslant c_1 (1+t)^{-a+\eta}\,. \tag{4.4}$$

This establishes (3.6) in the present situation. To estimate $\|H_0(g_t - h_t)\|$, one can use (3.7) but write

$$\left\|[\varphi(H) - \varphi(H_0)]g_t\right\| + \left\|V\psi(H)g_t\right\| \leqslant \left\{ \left\|[\varphi(H) - \varphi(H_0)]\langle Q\rangle^a\right\| + \left\|V\psi(H)\langle Q\rangle^a\right\| \right\} \left\|\langle Q\rangle^{-a} g_t\right\|\,.$$

The first factor on the R. H. S. is finite by Proposition 1(b,c), and

$$\begin{aligned}\left\|\langle Q\rangle^{-a} g_t\right\| &\leqslant \left\|\langle Q\rangle^{-a}\psi(H)\,U_t\,\Omega_+\psi(H_0)\langle Q\rangle^{-a}\right\|\left\|\langle Q\rangle^{a}h\right\| \\ &\leqslant \sup_{s>0}\left\|\langle Q\rangle^{-a}\psi(H)\,U_{t-s}\,U_s^0\,\psi(H_0)\langle Q\rangle^{-a}\right\|\left\|\langle Q\rangle^{a}h\right\| \\ &\leqslant c_2(1+|t|)^{-a+\eta}\left\|\langle Q\rangle^{a}h\right\|\end{aligned}$$

by Lemma 9(b) of [9].

Appendix

We indicate the proof of (2.5) with $\phi^0_{\Sigma_R}(h,t) \equiv \Phi_{\Sigma_R}(h_t)$, where

$$h_t = U_t^0 e^{-iX_t(\boldsymbol{P})}h$$

and X_t is a real-valued function of class C^{n+1} on $\mathbb{R}^n$ satisfying, for each $b>0$[†]:

$$K_b \equiv \max_{0\leqslant|\alpha|\leqslant n+1}\ \sup_{t\in\mathbb{R}}\ \sup_{k\geqslant b}|(\partial^\alpha X_t)(\boldsymbol{k})| < \infty\,. \tag{A.1}$$

We refer to *Case 1* if $X_t\equiv 0$ (this covers the situation of Section 3) and to *Case 2* if $X_t \not\equiv 0$. In *Case 1* we assume that $h\in\mathscr{D}_\mu$ for some $\mu > \frac{n}{2} + [\![\frac{n+1}{2}]\!]$, and in *Case 2* we require that $\tilde{h}\in C_0^{n+1}(\mathbb{R}^n\backslash\{0\})$. We set

$$h^t = e^{-iX_t(\boldsymbol{P})}h\,,$$

and we shall use the Dollard decomposition of the free evolution:

$$U_t^0 = Z_t\,G_t\,Z_t \tag{A.2}$$

where

$$[Z_t f](\boldsymbol{x}) = e^{i\frac{\boldsymbol{x}^2}{4t}}f(\boldsymbol{x})\,, \qquad [G_t f](\boldsymbol{x}) = (2it)^{-n/2}\tilde{f}(\tfrac{\boldsymbol{x}}{2t})\,. \tag{A.3}$$

The basic ideas of the proof are the same as in [7].

By part (i) of the proof of Proposition 2 or its modification mentioned in Section 4, we have

$$\left|\left[U_t^0 e^{-iX_t(\boldsymbol{P})}h\right](\boldsymbol{x})\right| \leqslant c\,(1+|t|)^m(1+r)^{-m} \quad \text{for } m=0,1,2,\dots,n.$$

This implies that

$$\left|\Phi_{\Sigma_R}(h_t)\right| \leqslant c_1(1+|t|)^n(1+R)^{-1}\,,$$

hence for any fixed T_1 and T_2 ($T_1<T_2$):

$$\int_{T_1}^{T_2}\left|\Phi_{\Sigma_R}(h_t)\right|dt \leqslant \frac{c_2}{1+R}\,.$$

[†]In this Appendix we set $k=|\boldsymbol{k}|$.

So it is enough to prove (2.5) for $T = 1$.

Since $\widetilde{h}$ has compact support in $\mathbb{R}^n$, h^t is of class C^∞. Moreover, for each multi-index β there is a constant $c(\beta)$ such that

$$\sup_{t\in\mathbb{R}} \left\|(1+r)^\rho \partial^\beta h^t(\boldsymbol{x})\right\|_{L^2(\mathbb{R}^n)} \leqslant c(\beta) \tag{A.4}$$

for $\rho \leqslant \mu$ in *Case 1* and for $\rho \leqslant n+1$ in *Case 2* . In *Case 1* this holds because $h^t \equiv h \in \mathscr{D}_\mu$ (observe that $\partial^\beta h \equiv i^{|\beta|}\boldsymbol{P}^\beta h \in \mathscr{D}_\mu$ also). To check (A.4) in *Case 2* , write

$$i^{-|\beta|}(2\pi)^{n/2}\left(1+r^{n+1}\right)\partial^\beta h^t(\boldsymbol{x}) = \int d^nk\, e^{i\boldsymbol{k}\cdot\boldsymbol{x}} e^{-iX_t(\boldsymbol{k})}\boldsymbol{k}^\beta \widetilde{h}(\boldsymbol{k}) + r^{n+1}\int d^nk\, e^{i\boldsymbol{k}\cdot\boldsymbol{x}} e^{-iX_t(\boldsymbol{k})}\boldsymbol{k}^\beta \widetilde{h}(\boldsymbol{k}).$$

In the last integral, we use the identity ($\boldsymbol{x} \neq \boldsymbol{0}$):

$$e^{i\boldsymbol{k}\cdot\boldsymbol{x}} = (-i)^\ell r^{-\ell}\left(\frac{\boldsymbol{x}}{r}\cdot\boldsymbol{\nabla}_{\boldsymbol{k}}\right)^\ell e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \tag{A.5}$$

(with $\ell = n+1$) and integrate by parts. If $b > 0$ is such that $\widetilde{h}(\boldsymbol{k}) = 0$ for $k \leqslant b$, then one obtains (A.4) with

$$c(\beta) = c_3(K_b+1)\sum_{|\alpha|\leqslant n+1}\left\|\partial^\alpha \boldsymbol{k}^\beta \widetilde{h}(\boldsymbol{k})\right\|_{L^2(\mathbb{R}^n)}.$$

The following consequence of (A.4) will be useful:

$$\sup_{t\in\mathbb{R}}\left\|(1+r)^\nu \partial^\beta h^t(\boldsymbol{x})\right\|_{L^1(\mathbb{R}^n)} \leqslant c'(\beta) \qquad \forall\, \nu \leqslant [\![\tfrac{n+1}{2}]\!]. \tag{A.6}$$

Now define (with $\mathscr{F}$ the Fourier transformation and $j = 1,\dots,n$):

$$\theta(h;\boldsymbol{k},t) = (2\pi)^{-n/2}\int d^nx\, e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} e^{i\frac{\boldsymbol{x}^2}{4t}} h^t(\boldsymbol{x}) = [\mathscr{F}Z_t h^t](\boldsymbol{k}) \tag{A.7}$$

$$\theta_j(h;\boldsymbol{k},t) = (2\pi)^{-n/2}\int d^nx\, e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} e^{i\frac{\boldsymbol{x}^2}{4t}} x_j h^t(\boldsymbol{x}) = [\mathscr{F}Z_t Q_j h^t](\boldsymbol{k}) \tag{A.8}$$

$$\eta(h;\boldsymbol{k},t) = (2\pi)^{-n/2}\int d^nx\, e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}\left(e^{i\frac{\boldsymbol{x}^2}{4t}} - 1\right)h^t(\boldsymbol{x}) = [\mathscr{F}(Z_t - I)h^t](\boldsymbol{k}). \tag{A.9}$$

We shall need the estimates

$$\sup_{t\geqslant 1}\sup_{\boldsymbol{k}\in\mathbb{R}^n}\left[\left|k^m\theta(h;\boldsymbol{k},t)\right| + \left|k^{m-1}\theta_j(h;\boldsymbol{k},t)\right| + t^{1/2}\left|k^m\eta(h;\boldsymbol{k},t)\right|\right] \leqslant c < \infty \tag{A.10}$$

valid for $m = 1,2,\dots,[\![\frac{n+1}{2}]\!]$. The proof of (A.10) is similar to that of (A.4); for example

$$k^m\eta(h;\boldsymbol{k},t) = i^m(2\pi)^{-n/2}\int d^nx\left(e^{i\frac{\boldsymbol{x}^2}{4t}} - 1\right)h^t(\boldsymbol{x})\left(\tfrac{\boldsymbol{k}}{k}\cdot\boldsymbol{\nabla}_{\boldsymbol{x}}\right)^m e^{-i\boldsymbol{k}\cdot\boldsymbol{x}},$$

and the last estimate in (A.10) is obtained, after integrating by parts, by taking into account (A.6) and the inequalities

$$\left|e^{i\frac{\boldsymbol{x}^2}{4t}}-1\right| \leqslant |t|^{-1/2}r$$

$$\left|\partial^\alpha e^{i\frac{\boldsymbol{x}^2}{4t}}\right| \leqslant c\,t^{-1}(1+r)^{|\alpha|} \qquad\qquad \text{if } 1\leqslant|\alpha|\leqslant m \text{ and } t\geqslant 1\,.$$

We also observe that, by the Riemann-Lebesgue lemma:

$$\lim_{k\to\infty}\Big[\big|k^m\theta(h;\boldsymbol{k},t)\big|+\big|k^m\eta(h;\boldsymbol{k},t)\big|\Big]=0 \qquad\qquad \text{for } m=0,1,2,\dots,[\![\tfrac{n+1}{2}]\!]\,. \tag{A.11}$$

We now use (A.2). Since $[P_j,Z_t]=\frac{1}{2t}Z_tQ_j$, one has $Q_jP_jU_t^0=Z_t[Q_jP_jG_tZ_t+\frac{1}{2t}Q_j^2G_tZ_t]$. Hence

$$\begin{aligned}\mathscr{J}(R) \equiv \int_1^\infty \Phi_{\Sigma_R}(h_t)\,dt &= 2\,\mathrm{Re}\int_1^\infty dt\int_{\Sigma_1}R^{n-1}\,d\omega\;\overline{\big[U_t^0h^t\big](R\boldsymbol{\omega})}\,\sum_{j=1}^n\big[\tfrac{Q_j}{R}\,P_j\,U_t^0h^t\big](R\boldsymbol{\omega})\\ &= \mathscr{J}_1(R)+\mathscr{J}_2(R)\,,\end{aligned}$$

with

$$\begin{aligned}\mathscr{J}_1(R) &= 2\,\mathrm{Re}\int_1^\infty dt\int_{\Sigma_1}R^{n-2}\,d\omega\;\overline{\big[G_tZ_th^t\big](R\boldsymbol{\omega})}\;\big[\boldsymbol{Q}\cdot\boldsymbol{P}\,G_tZ_th^t\big](R\boldsymbol{\omega})\,,\\ \mathscr{J}_2(R) &= \int_1^\infty\frac{dt}{t}\int_{\Sigma_1}R^n\,d\omega\Big|\big[G_tZ_th^t\big](R\boldsymbol{\omega})\Big|^2.\end{aligned}$$

To estimate $\mathscr{J}_1(R)$, we remark that

$$\begin{aligned}[G_tZ_th^t](\boldsymbol{x}) &= (2it)^{-n/2}\,\theta(h;\tfrac{\boldsymbol{x}}{2t},t)\\ [Q_jP_jG_tZ_th^t](\boldsymbol{x}) &= -\,(2it)^{-n/2}(2t)^{-1}x_j\theta_j(h;\tfrac{\boldsymbol{x}}{2t},t)\,,\end{aligned}$$

so that

$$\big|\mathscr{J}_1(R)\big| \leqslant 2\sum_{j=1}^n\int_1^\infty\frac{dt}{(2t)^2}\int_{S_1}d\omega\Big|\big(\tfrac{R}{2t}\big)^{\frac{n}{2}}\,\theta(h;\tfrac{R\omega}{2t},t)\Big|\Big|\big(\tfrac{R}{2t}\big)^{\frac{n}{2}-1}\,\theta_j(h;\tfrac{R\omega}{2t},t)\Big|\,.$$

By (A.10) the integrand is bounded by a constant independent of R and t, and by (A.10) and (A.11) it converges to zero as $R\to\infty$. Hence

$$\lim_{R\to\infty}\mathscr{J}_1(R)=0\,. \tag{A.12}$$

In $\mathscr{J}_2(R)$ we write $G_tZ_t=G_t+G_t(Z_t-I)$. Then

$$\begin{aligned}\mathscr{J}_2(R) = &\int_1^\infty\frac{dt}{t}\int_{\Sigma_1}R^n\,d\omega\,\frac{1}{(2t)^n}\,\Big|[\mathscr{F}h^t]\big(\tfrac{R\omega}{2t}\big)\Big|^2\\ &+\int_1^\infty\frac{dt}{t}\int_{\Sigma_1}R^n\,d\omega\,\frac{1}{(2t)^n}\,\Big|\eta(h;\tfrac{R\omega}{2t},t)\Big|^2\\ &+2\,\mathrm{Re}\int_1^\infty\frac{dt}{t}\int_{\Sigma_1}R^n\,d\omega\,\frac{1}{(2t)^n}\,\overline{[\mathscr{F}h^t]\big(\tfrac{R\omega}{2t}\big)}\;\eta(h;\tfrac{R\omega}{2t},t)\,.\end{aligned} \tag{A.13}$$

The last two terms converge to zero as $R \to \infty$, as can be seen by an argument similar to that given for $\mathscr{J}_1(R)$ (in both terms, use the fact that $\left|\left(\frac{R}{2t}\right)^{n/2} \eta(h; \frac{R\omega}{2t}, t)\right| \leqslant c\, t^{-1/2}$ and converges to zero (for fixed t) as $R \to \infty$; in the last term observe that $|[\mathscr{F} h^t](\boldsymbol{y})| = |\tilde{h}(\boldsymbol{y})|$, and that $\tilde{h}$ has compact support in $\mathbb{R}^n$). For the first term of (A.13) one makes the change of variables $t \mapsto k = \frac{R}{2t}$, which leads to

$$\int_1^\infty \frac{dt}{t} \int_{\Sigma_1} R^n \, d\omega \, \frac{1}{(2t)^n} \left|[\mathscr{F} h^t]\left(\tfrac{R\omega}{2t}\right)\right|^2 = \int_0^{\frac{R}{2}} k^{n-1} dk \int_{\Sigma_1} d\omega \, |\tilde{h}(k\boldsymbol{\omega})|^2 .$$

So

$$\lim_{R \to \infty} \mathscr{J}_2(R) = \int_C d^n k \, |\tilde{h}(\boldsymbol{k})|^2 . \tag{A.14}$$

This proves the equality of the first and the third term in (2.5). A simple adaptation of the preceding arguments leads to the equality of the second and the third term in (2.5).

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