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A semi-classical relativistic black hole[1]

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Abstract. A new two-dimensional black hole model, based on the "$R = T$" relativistic theory, is introduced, and the quantum massless scalar field is studied in its classical gravitational field. In particular infrared questions are discussed. The two-point function, energy-momentum tensor, current, Bogoliubov transformations and the mean number of created particles for a given test function are computed. I show that this black hole emits massless scalar particles spontaneously. Comparison with the corresponding field theory in a thermal bath shows that the spontaneous emission is *everywhere thermal*, i.e. not only near the horizon.

1 Introduction

S.W. Hawking discovered that, due to quantum mechanical effects, black holes spontaneously create and emit particles in 1+3 dimensions. He showed furthermore that the mean number of spontaneously created particles is thermal near the event-horizon [1]. The two-point function and the energy-momentum tensor of quantum matter were also computed in the gravitational field of black holes by other authors and their thermal properties studied [2, 3]. From these results it has been concluded that, near the event-horizon, the radiation of a 1+3 dimensional black hole is indeed thermal, with temperature inversely proportional to the mass.

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Recently there has been renewed interest in the study of 1+1 dimensional black hole models [4, 5], for which the technical difficulties encountered are of less importance than in the 1+3 dimensional case. In the present paper I investigate the semi-classical properties of a new 1+1 dimensional black hole model, based on the "$R = T$" theory. This theory was introduced by R. B. Mann [6]. The scalar curvature which defines this model vanishes everywhere, except on a light-like straight line where it is infinite and from which the horizon originates. I show that this infinite and localized curvature induces an emission of massless scalar particles which is thermal everywhere, i.e. not only near the horizon, and that the temperature of the radiation is proportional to the relative amplitude of the curvature.

In section 2 the "$R = T$" theory is reviewed and the new black hole model is introduced. In section 3 the quantization of the massless scalar field theory is reviewed in 1+1 dimensional Minkowski space-time. The quantization is extended to curved space-times in section 4, where it is also shown that the two-dimensional massless scalar field theory may be reduced to two independent one-dimensional scalar field theories under some specified conditions. Section 5 is devoted to the formal study of one-dimensional field theories obtained in this way. Relevant observables for the massless scalar field are introduced in section 6. Section 7 is devoted to the study of one-dimensional massless scalar field theories in a thermal bath. The results obtained are finally applied to the new black hole model in section 8.

2 The relativistic black hole model

The classical Einstein equations for the gravitational field are given by

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \; = \; 8\pi \mathcal{G}\, T_{\mu\nu}, \tag{2.1}$$

where $\mathcal{G}$ is the universal gravity constant and $c = 1$. They imply the covariant conservation of the classical energy-momentum tensor $T_{\mu\nu}$:

$$\nabla^{\mu} T_{\mu\nu} \; = \; 0. \tag{2.2}$$

The l.h.s. of eq. (2.1) vanishes for all 1+1 dimensional metrics, so that curvature is arbitrary and matter is excluded from 1+1 dimensional space-times [7]. In consequence the Einstein equations have no physical contents in two dimensions.

In spite of this fact, R.B. Mann [6] has extracted a non-trivial theory of gravity from the Einstein equations by considering the limit $D \rightarrow 2^{+}$, where the space-time dimension D is allowed to take continuum values. The trace of eq. (2.1) is given by

$$\left(1 - \frac{D}{2}\right) R(x) \; = \; 8\pi \mathcal{G}\, T(x), \tag{2.3}$$

using $g^{\mu\nu} g_{\mu\nu} = D$. Assuming that the constant $\mathcal{G}$ depends on the space-time dimension D and that the limit

$$\lim_{D \rightarrow 2^{+}} \frac{\mathcal{G}}{1 - D/2} \; = \; G \tag{2.4}$$

exists, then equation (2.3) implies

$$R(x) \;=\; 8\pi G\, T(x), \tag{2.5}$$

where $T(x) = T^{\mu}{}_{\mu}(x)$ is the trace of the energy-momentum tensor. Equation (2.5) does not imply the covariant conservation of $T_{\mu\nu}(x)$, so eq. (2.2) has to be imposed by hand.

For the trace $T(x)$ Mann et al. [5] have considered the form

$$T(x) \;=\; \frac{M}{8\pi G}\,\delta(x^1 - x^1_o), \tag{2.6}$$

and have shown that eqs (2.5) and (2.6) admit eternal black holes with a pair of horizons as solutions.

I assume now that $T(x)$ is given by

$$T(x) \;=\; \frac{M}{8\pi G}\,\delta(x^+ - x^+_o), \tag{2.7}$$

where $x^{\pm} = (x^0 \pm x^1)/\sqrt{2}$ and the constant M is strictly positive. Equation (2.7) is consistent with eq. (2.2) and describes a pulse of classical matter traveling with the velocity of light towards the left at $x^+ = x^+_o$. From eqs (2.5) and (2.7) the scalar curvature is given by

$$R(x) \;=\; 4M\;\delta(x^+ - x^+_o). \tag{2.8}$$

This equation defines a black hole model, as shown below, and is solved in the conformal gauge

$$ds^2 \;=\; C(x)\; dx^+ dx^-. \tag{2.9}$$

Equation (2.8) implies that the conformal factor $C(x)$ satisfies the non-linear equation

$$\partial_+\partial_- \log|C(x)| \;=\; M\, C(x)\, \delta(x^+ - x^+_o). \tag{2.10}$$

This may be rewritten as:

$$\partial_- \log|C(x)| \;=\; \begin{cases} C_o, & \text{if } x^+ < x^+_o, \\ M\, C(x^+_o, x^-) + C_o, & \text{if } x^+ > x^+_o, \end{cases} \tag{2.11}$$

where C_o is a real constant, which shows that the metric is modified at $x^+ = x^+_o$ by the pulse of matter. This last equation implies that the conformal factor $C(x)$ depends only on x^- in the half-plane $x^+ > x^+_o$, and that this is discontinuous at $x^+ = x^+_o$. This discontinuity comes from the singularity of the curvature (2.8) at this same value of x^+ and it may be removed by replacing the delta function (2.7) by a sharp continuous pulse centered in a neighborhood of $x^+ = x^+_o$. It is easy to check that a solution of eq. (2.11) for $C_o = 0$ is given by

$$ds^2 \;=\; \begin{cases} dx^+\, dx^-, & \text{if } x^+ < x^+_o, \\ \dfrac{dx^+\, dx^-}{M\,(\Delta - x^-)}, & \text{if } x^+ > x^+_o, \end{cases} \tag{2.12}$$

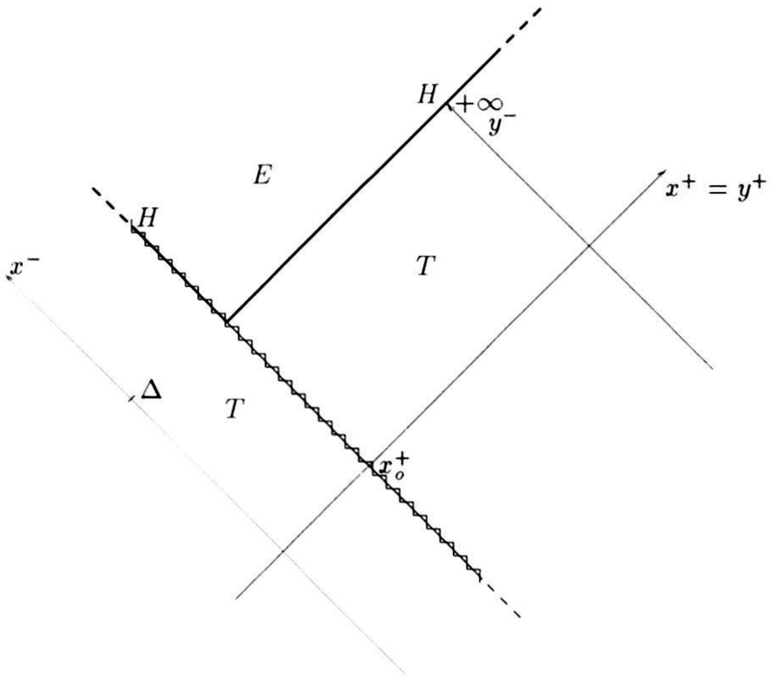

Figure 1: The space-time structure of the relativistic black hole. The broken line denotes the curvature singularity superimposed on the pulse of matter; T and E are the time-like and space-time regions respectively; H is the horizon.

where Δ is an arbitrary constant reflecting the invariance of curvature (2.8) under translations of x^-. Note that to obtain this solution the continuity of $C(x)^{-1}$ has been required at $x^- = \Delta$ and $x^+ > x_o^+$, where the metric is singular.

In a given set of conformal coordinates the *horizon* will be defined as the curve where the metric reverses its sign. It thus divides space-time into a *time-like* and a *space-like region*, where the conformal factor is positive and negative respectively. The value of the metric may be null or singular on the horizon. In our case it is singular. The horizon associated with the metric (2.12) is made up of (see figure 1):

- a half-straight line defined by $x^+ \geq x_o^+$ and $x^- = \Delta$ which originates from the singularity of the curvature;
- a half-straight line defined by $x^+ = x_o^+$ and $x^- \geq \Delta$ superimposed on the singularity of the curvature.

The space-like region is identified as the interior of a black hole, since the events located in it are not in the past of any observer situated in the flat part of the time-like region for all times. This black hole will be called a *relativistic black hole*, because it is based on the relativistic equation (2.5).

Since the coordinates $(x^+, x^-) \in \mathrm{I\!R}^2$ are Minkowskian in the "past" half-plane M_P defined by (see eq. (2.12))

$$M_P = \{ x \in \mathrm{I\!R}^2 \mid x^+ < x_o^+ \}, \tag{2.13}$$

they will be called *incoming coordinates.* Another set of conformal coordinates $(y^+, y^-) \in \mathbb{R}^2$ is defined by the transformation

$$\begin{cases} x^+(y^+) &= y^+, \\ x^-(y^-) &= \Delta - e^{-My^-}, \end{cases} \tag{2.14}$$

which satisfies:

$$\lim_{y^- \to +\infty} x^-(y^-) = \Delta, \tag{2.15}$$

$$\lim_{y^- \to -\infty} x^-(y^-) = -\infty. \tag{2.16}$$

The horizon is located at $y^- = +\infty$ in the new coordinates. These coordinates cover only the lower part of space-time R defined by

$$R = \{ x \in \mathbb{R}^2 \mid x^- < \Delta \}, \tag{2.17}$$

where the metric (2.12) is given by

$$ds^2 = \begin{cases} M\, e^{-My^-}\, dy^+\, dy^-, & \text{if } y^+ < y_o^+, \\ dy^+\, dy^-, & \text{if } y^+ > y_o^+, \end{cases} \tag{2.18}$$

where $y_o^+ = x_o^+$. Since the coordinates (y^+, y^-) are Minkowskian in the "future" half-plane M_F defined by

$$M_F = \{ y \in \mathbb{R}^2 \mid y^+ > y_o^+ \}, \tag{2.19}$$

they will be called *outgoing coordinates.*

The transformation (2.14), which relates incoming and outgoing coordinates, is intimately related to the space-time structure. It will play an important role in the analysis of the black hole semi-classical properties. Note that the right transformation $x^-(y^-)$ may be extended analytically in the whole complex plane and that it exhibits an imaginary period given by $\frac{2\pi}{M}$ for all the values of its argument:

$$x^-(y^-) = x^-\left(y^- + \mathrm{i}\frac{2\pi}{M} n\right), \qquad \forall n \in \mathbb{Z}, \forall y^- \in \mathbb{R}. \tag{2.20}$$

This period will turn out to be the inverse temperature β of the black hole radiation.

3 Quantization of the massless scalar field

Before considering the quantum physics of the massless scalar field in 2D curved space-times, its quantization in 2D Minkowski space-time should be reviewed. This cannot be carried out by imposing all the Wightman axioms [8] in a standard way. In particular the positivity of the Wightman function cannot be satisfied for all Schwartz test functions because of its bad infrared behavior. Consequently either the massless scalar field should be quantized in an indefinite metric following G. Morchio et al. [9], or the space of test functions should be restricted in order to satisfy the positivity condition, as proposed by S. Fulling and S. Ruijsenaars [10]. For simplicity I will adopt the second point of view.

In the 2D Minkowski space-time the *Wightman distribution* of the massless scalar field is defined on the Schwartz space $\mathcal{S}(\mathrm{IR}^2)$ by [11]

$$W_o[h_1 \times h_2^*] = \int_{\mathrm{IR}^2} d^2k\, \widetilde{W}_o(k)\, \tilde{h}_1(k)\, \tilde{h}_2(k)^*, \tag{3.1}$$

where

$$\widetilde{W}_o(k) = \frac{1}{2}\left\{ \delta(k_-)\, \frac{d}{dk_+}[\theta(k_+)\log k_+] + \delta(k_+)\, \frac{d}{dk_-}[\theta(k_-)\log k_-] \right\}. \tag{3.2}$$

Performing a 2D Fourier transform[3], the Wightman distribution may also be expressed in the form

$$W_o[h_1 \times h_2^*] = \int_{\mathrm{IR}^2} d^2x \int_{\mathrm{IR}^2} d^2\bar{x}\, h_1(x)\, W_o(x,\bar{x})\, h_2(\bar{x})^*, \tag{3.3}$$

where $W_o(x,\bar{x}) = W_o(x-\bar{x})$ is the *Wightman function* and is given by

$$W_o(x) = -\frac{1}{8\pi^2}\log\left(-x^2 + \mathrm{i}x^0 0^+\right) - \frac{\gamma}{4\pi^2}, \tag{3.4}$$

where γ is the Euler constant. The Wightman function (3.4) satisfies *i)* the covariance property, $W_o(\Lambda x) = W_o(x)$ for any Lorentz transformation Λ; *ii)* the spectral condition, $\widetilde{W}_o(k) = 0$ if $k^2 < 0$; *iii)* the locality property, $W_o(x) = W_o(-x)$ if $x^2 < 0$. However the positivity condition, $W_o[h \times h^*] \geq 0\ \forall h \in \mathcal{S}(\mathrm{IR}^2)$, is not generally satisfied (consider $\tilde{h}(k) = e^{-\alpha k^2}$). In consequence a standard quantum relativistic interpretation of the theory is not possible.

To elude this difficulty, the function space is restricted to all Schwartz functions vanishing for null momentum. The test function space $\mathcal{S}_0(\mathrm{IR}^2)$ is defined by

$$\mathcal{S}_0(\mathrm{IR}^2) = \{\tilde{h} \in \mathcal{S}(\mathrm{IR}^2) \mid \tilde{h}(0) = 0\}, \tag{3.5}$$

and the Wightman distribution (3.1) restricted to this space function is given by

$$W_o[h_1 \times h_2^*] = \int_{-\infty}^{+\infty} \frac{dk_1}{2|k_1|}\left[\tilde{h}_1(k)\,\tilde{h}_2(k)^*\right]_{k_o = |k_1|}, \tag{3.6}$$

where $\tilde{h}_1, \tilde{h}_2 \in \mathcal{S}_0(\mathrm{IR}^2)$. This clearly satisfies the positivity condition and thus defines a scalar product on $\mathcal{S}_0(\mathrm{IR}^2)$. A restricted Hilbert space $\mathcal{H}$ may now be constructed from the

[3]The 2D Fourier transform is defined by $\tilde{h}(k) = \frac{1}{2\pi}\int_{\mathrm{IR}^2} d^2x\, h(x)\, e^{-\mathrm{i}k\cdot x}$.

Wightman distribution (3.6), which is related to the two-point function of the scalar field ϕ by

$$(\Omega_o, \phi[h_1]\, \phi[h_2]^\dagger\, \Omega_o) \;=\; W_o[\, h_1 \times h_2^* \,], \tag{3.7}$$

where Ω_o is the vacuum of $\mathcal{H}$.

In 2D Minkowski space-time the scalar field $\phi(x)$ satisfies the massless Klein-Gordon equation:

$$\frac{\partial^2 \phi}{\partial x^+ \, \partial x^-} \;=\; 0. \tag{3.8}$$

Its general solution will be written in the form

$$\phi(x) \;=\; \frac{1}{\sqrt{2\pi}} \left[\, \phi_+(x^+) \;+\; \phi_-(x^-) \,\right], \tag{3.9}$$

where $\phi_+(x^+)$ and $\phi_-(x^-)$ are the left and right moving fields. These will be called *1D fields*, in opposition to $\phi(x)$ which is a *2D field*.

The quantum scalar field ϕ is defined as a distribution by

$$\phi[h] \;=\; \int_{\mathbb{R}^2} d^2x \, \phi(x)\, h(x), \tag{3.10}$$

where h is any *2D test function* belonging to $\mathcal{S}_0(\mathbb{R}^2)$. The *1D test functions* $h_\pm$ are constructed from the test function h by integrating on $x^\mp$:

$$h_\pm(x^\pm) \;=\; \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dx^\mp \;\, h(x). \tag{3.11}$$

The Fourier transforms of h and $h_\pm$ are related by[4]

$$\tilde{h}_\pm(k_\mp) \;=\; \tilde{h}(k)\Big|_{k_\pm = 0}, \tag{3.12}$$

and this shows that the functions $\tilde{h}_\pm$ belong to the *1D test function space* $\mathcal{S}_0(\mathbb{R})$ defined by

$$\mathcal{S}_0(\mathbb{R}) \;=\; \{\, \tilde{h}_\pm \in \mathcal{S}(\mathbb{R}) \mid \tilde{h}_\pm(0) = 0 \,\}, \tag{3.13}$$

if $\tilde{h} \in \mathcal{S}_0(\mathbb{R}^2)$. The *1D scalar field distributions* are defined by

$$\phi_\pm[h_\pm] \;=\; \int_{-\infty}^{+\infty} dx^\pm \; \phi_\pm(x^\pm)\, h_\pm(x^\pm), \tag{3.14}$$

where $\tilde{h}_\pm \in \mathcal{S}_0(\mathbb{R})$. From the previous definitions we deduce that the 2D field distribution (3.10) is equal to the sum of the 1D field distributions (3.14):

$$\phi[h] \;=\; \phi_+[h_+] \;+\; \phi_-[h_-]. \tag{3.15}$$

[4]The 1D Fourier transform is defined by $\tilde{h}_\pm(k_\mp) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} dx^\pm \, h_\pm(x^\pm)\, e^{-ik_\mp x^\pm}$.

The *1D Wightman distributions* will be defined on $\mathcal{S}_0(\mathbb{R}) \times \mathcal{S}_0(\mathbb{R})$ by

$$W_o^{\pm}[\,h_{1\pm} \times h_{2\pm}^*\,] \;=\; \int_{-\infty}^{+\infty} dx^{\pm} \int_{-\infty}^{+\infty} d\bar{x}^{\pm}\; h_{1\pm}(x^{\pm})\; W_o^{\pm}(x^{\pm} - \bar{x}^{\pm})\; h_{2\pm}(\bar{x}^{\pm})^*, \tag{3.16}$$

where the *1D Wightman functions* $W_o^{\pm}(x^{\pm}, \bar{x}^{\pm}) = W_o^{\pm}(x^{\pm} - \bar{x}^{\pm})$ are given, up to a constant, by

$$W_o^{\pm}(x^{\pm} - \bar{x}^{\pm}) \;=\; -\frac{1}{4\pi} \log\left(\bar{x}^{\pm} - x^{\pm} + \mathrm{i}0^+\right). \tag{3.17}$$

From these definitions and eq. (3.4) we deduce that the 2D Wightman distribution (3.3) is equal to the sum of the 1D Wightman distributions (3.16)

$$W_o[\,h_1 \times h_2^*\,] \;=\; W_o^{+}[\,h_{1+} \times h_{2+}^*\,] \;+\; W_o^{-}[\,h_{1-} \times h_{2-}^*\,], \tag{3.18}$$

which are also given by

$$W_o^{\pm}[\,h_{1\pm} \times h_{2\pm}^*\,] \;=\; \int_0^{\infty} \frac{dk_{\mp}}{2k_{\mp}}\; \tilde{h}_{1\pm}(k_{\mp})\; \tilde{h}_{2\pm}(k_{\mp})^*, \tag{3.19}$$

where $\tilde{h}_1, \tilde{h}_2 \in \mathcal{S}_0(\mathbb{R})$. These are related to the two-point functions by the equations

$$(\Omega_o, \phi(x)\,\phi(\bar{x})^{\dagger}\,\Omega_o) \;=\; W_o(x - \bar{x}), \tag{3.20}$$

$$(\Omega_o, \phi_{\pm}(x^{\pm})\,\phi_{\pm}(\bar{x}^{\pm})^{\dagger}\,\Omega_o) \;=\; W_o^{\pm}(x^{\pm} - \bar{x}^{\pm}), \tag{3.21}$$

$$(\Omega_o, \phi_{\pm}(x^{\pm})\,\phi_{\mp}(\bar{x}^{\mp})^{\dagger}\,\Omega_o) \;=\; 0, \tag{3.22}$$

from which the fields commutators are computed[5]:

$$\left[\phi(x),\, \phi(\bar{x})^{\dagger}\right] \;=\; \frac{\mathrm{i}}{4\pi}\,\theta[\,(\bar{x} - x)^2\,]\;\mathrm{sgn}\,(\bar{x}^0 - x^0), \tag{3.23}$$

$$\left[\phi_{\pm}(x^{\pm}),\, \phi_{\pm}(\bar{x}^{\pm})^{\dagger}\right] \;=\; \frac{\mathrm{i}}{4}\,\mathrm{sgn}\,(\bar{x}^{\pm} - x^{\pm}), \tag{3.24}$$

$$\left[\phi_{+}(x^{+}),\, \phi_{-}(\bar{x}^{-})^{\dagger}\right] \;=\; 0. \tag{3.25}$$

Equations (3.15), (3.18), (3.22) and (3.25) show that the 2D massless scalar field may be considered as two uncoupled right and left 1D fields.

We close this section by defining the notion of particle in one and two dimensions. These definitions will be useful below. The function $h \in \mathcal{S}_0(\mathbb{R}^2)$ is said to be a *2D particle test function* if

$$\tilde{h}(k)\Big|_{k_o = -|k_1|} \;=\; 0, \qquad \forall\; k_1 \in \mathbb{R}. \tag{3.26}$$

Similarly, the functions $h_{\pm} \in \mathcal{S}_0(\mathbb{R})$ are said to be *1D particle test functions* if

$$\tilde{h}_{\pm}(k^{\mp}) \;=\; 0, \qquad \forall\; k^{\mp} < 0. \tag{3.27}$$

In the 2D Minkowski space-time eq. (3.12) implies that these definitions are equivalent.

[5]The equality $2\theta(x^2)\,\mathrm{sgn}\,x^0 = \mathrm{sgn}\,x^+ + \mathrm{sgn}\,x^-$ is used to obtain eq. (3.24) from eq. (3.23).

4 The massless scalar field in curved space-times

In this section the field distribution in curved space-times is introduced and the relationship between field distributions in different coordinates is considered. The 1D field distributions are defined as in the 2D Minkowski space-time, whereas the 1D test functions are defined so as to take into account the metric. I show that, under specified conditions, the relationship between the 2D field distributions breaks down into two relationships between 1D field distributions, so that the 2D quantum problem is reduced to two independent 1D quantum problems. The particle and vacuum concepts are discussed for asymptotically Minkowskian coordinates at the end of this section.

I assume that the coordinates $x \in \mathbb{R}^2$ cover a whole 2D space-time. New coordinates y are introduced by the transformation

$$y \longrightarrow x(y), \qquad y \in \mathbb{R}^2, \tag{4.1}$$

and they will cover in general only a part R of space-time contained in the time-like region. The scalar fields $\phi(x)$ and $\hat{\phi}(y)$ in these coordinates will be called the *incoming* and *outgoing fields* respectively. They are related by

$$\hat{\phi}(y) = \phi(x(y)), \qquad \forall y \in \mathbb{R}^2. \tag{4.2}$$

The field distributions in both coordinates are defined as follows [12]:

$$\phi[h] = \int_{\mathbb{R}^2} d^2x \sqrt{-g(x)}\, \phi(x)\, h(x), \tag{4.3}$$

$$\hat{\phi}[f] = \int_{\mathbb{R}^2} d^2y \sqrt{-\hat{g}(y)}\, \hat{\phi}(y)\, f(y), \tag{4.4}$$

where $h, f \in \mathcal{S}_0(\mathbb{R}^2)$. These definitions are a generalization of eq. (3.10) to curved space-times. The determinants $g(x)$ and $\hat{g}(y)$ of the metric are related by

$$\hat{g}(y) = \left| \frac{\partial x}{\partial y}(y) \right|^2 g(x(y)), \qquad \forall y \in \mathbb{R}^2, \tag{4.5}$$

where $|\partial y/\partial x|$ is the Jacobian of the transformation (4.1).

Field distributions are considered as geometrical objects whose values do not depend on the coordinates chosen to express them. The distributions (4.3) and (4.4) are thus related by[6]

$$\hat{\phi}[f] = \phi[\hat{f}], \qquad \forall f \in \mathcal{S}_0(\mathbb{R}^2). \tag{4.6}$$

[6]Note that $f \in \mathcal{S}_0(\mathbb{R}^2)$ does not necessarily imply $\hat{f} \in \mathcal{S}_0(\mathbb{R}^2)$. If $\hat{f} \notin \mathcal{S}_0(\mathbb{R}^2)$, eq. (4.6) is only valid formally.

In the region R, this last equation defines the *incoming test function* $\hat{f}(x)$ in terms of the *outgoing test function* $f(y)$, and I will assume that $\hat{f}(x)$ vanishes outside the region R. Equations (4.2), (4.5) and (4.6) imply that these test functions are related in R by

$$f(y) \;=\; \hat{f}(x(y)), \qquad \forall y \in \mathbb{R}^2. \tag{4.7}$$

Assuming now that the coordinates x are conformal and that the transformation of coordinates $x = x(y)$ is given by

$$(y^+, y^-) \;\longrightarrow\; (x^+(y^+), x^-(y^-)), \qquad (y^+, y^-) \in \mathbb{R}^2, \tag{4.8}$$

then the coordinates y are also conformal. The property (4.8) is satisfied for the relativistic black hole model (see eq. (2.14)). In 2D curved space-times, the massless Klein-Gordon equation for conformal coordinates is formally identical to the one in 2D Minkowski space-time. Thus the incoming $\phi(x)$ and outgoing $\hat{\phi}(y)$ fields satisfy respectively eq. (3.8) and

$$\frac{\partial^2 \hat{\phi}}{\partial y^+ \, \partial y^-} \;=\; 0, \tag{4.9}$$

whose solutions are given by eq. (3.9) and

$$\hat{\phi}(y) \;=\; \frac{1}{\sqrt{2\pi}} \left[\hat{\phi}_+(y^+) \;+\; \hat{\phi}_-(y^-) \right]. \tag{4.10}$$

The relation between the left and right fields is deduced from eq. (4.2) up to a constant:

$$\hat{\phi}_\pm(y^\pm) \;=\; \phi_\pm(x^\pm(y^\pm)), \qquad \forall y^\pm \in \mathbb{R}. \tag{4.11}$$

In 2D curved space-times the *1D test functions* are defined by

$$h_\pm(x^\pm) \;=\; \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dx^\mp \sqrt{-g(x)}\; h(x), \tag{4.12}$$

$$f_\pm(y^\pm) \;=\; \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dy^\mp \sqrt{-\hat{g}(y)}\; f(y). \tag{4.13}$$

These definitions include the determinant of the metric and are a generalization of eq. (3.11). The *1D incoming and outgoing field distributions* are defined as in Minkowski space-time and are given by eq. (3.14) and

$$\hat{\phi}_\pm[f_\pm] \;=\; \int_{-\infty}^{+\infty} dy^\pm \, \phi_\pm(y^\pm)\, f_\pm(y^\pm). \tag{4.14}$$

Equation (3.15) is still valid in 2D curved space-times in the x and y coordinates:

$$\phi[h] \;=\; \phi_+[h_+] \;+\; \phi_-[h_-], \tag{4.15}$$

$$\hat{\phi}[f] \;=\; \hat{\phi}_+[f_+] \;+\; \hat{\phi}_-[f_-]. \tag{4.16}$$

The transformations for the 1D test functions are deduced from eqs (4.7), (4.12) and (4.13):

$$f_\pm(y^\pm) = \frac{\partial x^\pm}{\partial y^\pm}(y^\pm)\, \hat{f}_\pm(x^\pm(y^\pm)), \qquad y^\pm \in \mathbb{R}. \tag{4.17}$$

The metric does not appear explicitly in these transformations although they contain the dynamics of the problem. They imply that the 2D field transformation (4.6) may be broken down into two 1D left and right field transformations:

$$\hat{\phi}_+[f_+] = \phi_+[\hat{f}_+], \qquad \hat{\phi}_-[f_-] = \phi_-[\hat{f}_-]. \tag{4.18}$$

I must emphasize that the 1D field distributions $\hat{\phi}_\pm$ and $\phi_\pm$ are formally identical with their Minkowskian counterparts. Equations (4.18) imply that the left and right modes of the fields are not mixed up by changing coordinates. They are thus dynamically independent.

Note that the definitions (3.26) and (3.27) for 2D and 1D particle test functions are not *strictly* equivalent in curved space-times in any coordinates (see eq. (4.12) or (4.13)). There may however be *approximate* equivalence if the 2D test function is "well localized"[7] in a space-time region M where the metric is (asymptotically) Minkowskian. This shows that it is difficult to give a precise meaning to the notion of particle in curved space-time and in particular to make this meaning coincident with that of the Minkowskian field theory.

We note furthermore that the notions of particle are different in the x and y coordinates. In the 1D language, the particles test functions are defined respectively by

$$\tilde{h}_\pm(k^\pm) = 0, \qquad \text{if } k^\mp < 0, \tag{4.19}$$

$$\tilde{f}_\pm(p^\pm) = 0, \qquad \text{if } p^\mp < 0. \tag{4.20}$$

These conditions are incompatible unless the transformation $x(y)$ is the identity, i.e. the scalar curvature vanishes everywhere. This incompatibility is the key to understanding the creation of particles in curved space-times.

We assume from now on that the coordinates x and y are (asymptotically) Minkowskian in past and future space-time regions M_P and M_F respectively (as is the case in the relativistic black hole model). In consequence, they will be called *incoming* and *outgoing coordinates* respectively. If the test functions $h(x)$ and $f(y)$ are well localized in M_P and M_F, and satisfy respectively eqs (4.19) and (4.20), then they will respectively describe *incoming* and *outgoing particles.*

The *incoming and outgoing vacuums*, Ω_o and Ψ_o, will be defined in the 1D language by

$$\phi_\pm[h_\pm]\,\Omega_o = 0, \tag{4.21}$$

$$\hat{\phi}_\pm[f_\pm]\,\Psi_o = 0, \tag{4.22}$$

[7]Note that a 2D particle test function cannot in general be strictly localized, since its Fourier transform does not contain negative contributions.

where $h_\pm(x^\pm)$ and $f_\pm(y^\pm)$ are arbitrary 1D particle test functions (i.e. they satisfy respectively eqs (4.19) and (4.20)). Furthermore, if the corresponding 2D test functions $h(x)$ and $f(y)$ are also well localized in M_P and M_F respectively, these equations imply from eqs (4.15) and (4.16)

$$\phi[h]\,\Omega_o \approx 0, \tag{4.23}$$

$$\hat{\phi}[f]\,\Psi_o \approx 0, \tag{4.24}$$

and the functions $h(x)$ and $f(y)$ are 2D particle test functions (i.e. $h(x)$ satisfies eq. (3.26) in the incoming coordinates and $f(y)$ satisfies a similar equation in the outgoing coordinates). We thus conclude that the vacuums Ω_o and Ψ_o are ordinary Minkowskian vacuums. In particular, the incoming vacuum Ω_o is formally equivalent to the vacuum of the preceding section and consequently eqs (3.20) to (3.22) for the two-point functions are also valid in curved space-times.

5 One-dimensional scalar field theory

In this section the one dimensional scalar field theories are studied. I show that the commutation relations of the fields are invariant under any change of coordinates. The Bogoliubov transformations between the incoming and outgoing field operators are obtained and their implementability is discussed. Note that, for the relativistic black hole model, the physics of the left moving field ϕ_+ is trivial, since the transformation (2.14) between the left coordinates is the identity. I shall consider from now on only the right moving field ϕ_- and shall drop the subscript $-$.

The scalar product $\langle\ ,\ \rangle$ of two test functions is given by (see eq. (3.19))

$$\langle \tilde{f}_2, \tilde{f}_1 \rangle = \int_0^\infty \frac{dp}{2p}\, \tilde{f}_2(p)^*\, \tilde{f}_1(p), \tag{5.1}$$

where $\tilde{f}_1, \tilde{f}_2 \in \mathcal{S}_0(\mathbb{R})$. The norm $\|\ \|$ is defined by

$$\| \tilde{f} \|^2 = \langle \tilde{f}, \tilde{f} \rangle. \tag{5.2}$$

We define furthermore the function spaces

$$\mathcal{S}(\mathbb{R}_+) = \{\, \tilde{f} \in \mathcal{S}_0(\mathbb{R}) \mid \tilde{f}(p) = \theta(p)\tilde{f}(p) \ \ \forall p \in \mathbb{R} \,\}, \tag{5.3}$$

$$L^2(\tfrac{dp}{2p}, \mathbb{R}_+) = \{\, \tilde{f} \mid \tilde{f}(p) = \theta(p)\tilde{f}(p) \ \ \forall p \in \mathbb{R} \ \text{and} \ \|\tilde{f}\| < \infty \,\}. \tag{5.4}$$

Note that

$$\overline{\mathcal{S}(\mathbb{R}_+)}^{\|\ \|} = L^2(\tfrac{dp}{2p}, \mathbb{R}_+). \tag{5.5}$$

The set $\mathcal{S}_0(\mathbb{R})$ is the *particle test function space* and $L^2(\frac{dp}{2p}, \mathbb{R}_+)$ is the *particle wave function space.*

We recall that the incoming and outgoing test functions are related by (see eq. (4.17))

$$f(y) \;=\; \frac{\partial x}{\partial y}(y)\, \hat{f}(x(y)), \qquad \forall y \in \mathbb{R}. \tag{5.6}$$

It is not clear whether the inverse Fourier transform $f(y)$ and the Fourier transform $\tilde{\hat{f}}(k)$ exist if $\tilde{f} \in L^2(\frac{dp}{2p}, \mathbb{R}_+)$. For simplicity, I will assume in the following that $f(y)$ exists a.e. and is integrable, so that the existence of $\tilde{\hat{f}}(k)$ is certain. Note that $f \in L^1(dy, \mathbb{R})$ implies that $\hat{f}(x)$ is also integrable. This hypothesis is thus formulated in a way which is invariant under any transformation of coordinates. It also implies that the Fourier transforms $\tilde{f}(p)$ and $\tilde{\hat{f}}(k)$ are continuous everywhere and vanish at infinity. The incoming and outgoing momenta will be denoted by k and p respectively.

The Fourier transforms of the incoming and outgoing wave functions will be related by the operator U defined by

$$\tilde{\hat{f}}(k) \;=\; \int_0^\infty dp\, U(k,p)\, \tilde{f}(p), \tag{5.7}$$

whose kernel $U(k,p)$ is given by

$$U(k,p) \;=\; \frac{1}{2\pi} \int_{-\infty}^{+\infty} dy\; e^{-ikx(y)}\, e^{ipy}. \tag{5.8}$$

For any transformations $x = x(y)$, this satisfies the property

$$U(0,p) \;=\; \delta(p), \qquad \forall p \in \mathbb{R}_+, \tag{5.9}$$

which implies $\tilde{\hat{f}}(0) = \tilde{f}(0) = 0$ under our assumptions.

The positive and negative momentum components of the outgoing and incoming test functions $\tilde{f}(p)$ and $\tilde{\hat{f}}(k)$ are defined as

$$\begin{aligned} \tilde{f}_P(p) &= \theta(p)\,\tilde{f}(p), & \tilde{f}_N(p) &= \theta(p)\,\tilde{f}(-p), \\ \tilde{\hat{f}}_P(p) &= \theta(k)\,\tilde{\hat{f}}(k), & \tilde{\hat{f}}_N(p) &= \theta(k)\,\tilde{\hat{f}}(-k). \end{aligned} \tag{5.10}$$

The operators A and B will be defined respectively as the positive and negative incoming momentum contributions of U

$$(A\tilde{f})(k) \;=\; (U\tilde{f})_P(k), \qquad (B\tilde{f})(k) \;=\; (U\tilde{f})_N(k), \tag{5.11}$$

and the bilinear operator G by

$$G(f_1 \times f_2) \;=\; -\frac{1}{4\pi} \int_{-\infty}^{+\infty} dy \int_{-\infty}^{+\infty} dy'\, f_1(y) \log\left[\frac{x(y) - x(y')}{y - y'}\right] f_2(y'). \tag{5.12}$$

The logarithm in the integrand of this double integral is well defined since $x(y)$ is always an increasing function. The scalar product of incoming functions such as (5.11) may be expressed in terms of the bilinear operator G evaluated for the corresponding outgoing functions, as shown in the following theorem.

Theorem 1 *If $\tilde{f}_1, \tilde{f}_2 \in L^2(\frac{dp}{2p}, \mathbb{R}_+)$ are two wave functions such that their inverse Fourier transforms exist and are integrable, then*

$$\langle A\tilde{f}_2, A\tilde{f}_1 \rangle = G\,(f_1 \times f_2^*) + \langle \tilde{f}_2, \tilde{f}_1 \rangle, \tag{5.13}$$
$$\langle B\tilde{f}_2, B\tilde{f}_1 \rangle = G\,(f_1 \times f_2^*), \tag{5.14}$$
$$\langle A^*\tilde{f}_2^{\,*}, B\tilde{f}_1 \rangle = G\,(f_1 \times f_2), \tag{5.15}$$
$$\langle B^*\tilde{f}_2^{\,*}, A\tilde{f}_1 \rangle = G\,(f_1 \times f_2), \tag{5.16}$$

and hence[8]

$$A^\dagger A = B^\dagger B + E, \tag{5.17}$$
$$A^T B = B^T A, \tag{5.18}$$

where E is the identity.

Equation (5.14) is proved in appendix A.1 and the others results of this theorem are proved in a similar way.

We recall that the incoming and outgoing fields are related by (see eq. (4.11))

$$\hat{\phi}(y) = \phi(x(y)), \qquad \forall\, y \in \mathbb{R}. \tag{5.19}$$

The Wightman function for the incoming fields is given by the equation (see eq. (3.19))

$$(\Omega_o, \phi[h_1]\,\phi[h_2]^\dagger\,\Omega_o) = \int_0^\infty \frac{dk}{2k}\,\tilde{h}_2(k)^*\,\tilde{h}_1(k), \tag{5.20}$$

from which their commutator is deduced[9]

$$\left[\,\phi[h_1],\,\phi[h_2]^\dagger\,\right] = \int_{-\infty}^{+\infty} \frac{dk}{2k}\,\tilde{h}_2(k)^*\,\tilde{h}_2(k), \tag{5.21}$$

if $\tilde{h}_1, \tilde{h}_2 \in \mathcal{S}_0(\mathbb{R})$. We have a similar result for the outgoing fields. The equality

$$\int_{-\infty}^{+\infty} \frac{dk}{2k}\,\hat{\tilde{f}}_2(k)^*\,\hat{\tilde{f}}_1(k) = \int_{-\infty}^{+\infty} \frac{dp}{2p}\,\tilde{f}_2(p)^*\,\tilde{f}_1(p), \tag{5.22}$$

proved in appendix A.2, implies that the field commutator is invariant under any transformation of coordinates $x = x(y)$

$$\left[\,\hat{\phi}[f_1],\,\hat{\phi}[f_2]^\dagger\,\right] = \left[\,\phi[f_1],\,\phi[f_2]^\dagger\,\right], \tag{5.23}$$

where $\tilde{f}_1, \tilde{f}_2 \in \mathcal{S}_0(\mathbb{R})$. The Wightman function (5.20) is, however, not invariant under any non-trivial transformation of coordinates.

[8]Equations (5.17) and (5.18) were first obtained by R. M. Wald [13].

[9]Equation (5.21) may also be obtained from the commutator (3.24) using $\frac{i}{\pi}\int_{-\infty}^{+\infty} dk\, P\frac{1}{k}\, e^{-ikx} = \operatorname{sgn} x$.

In the *real* scalar fields, the incoming and outgoing field operators $a_{in,out}$ and $a^{\dagger}_{in,out}$ are defined by splitting the positive and negative momentum contributions of the incoming and outgoing fields:

$$\phi[h] = a_{in}[\tilde{h}_P] + a^{\dagger}_{in}[\tilde{h}_N], \tag{5.24}$$

$$\hat{\phi}[f] = a_{out}[\tilde{f}_P] + a^{\dagger}_{out}[\tilde{f}_N], \tag{5.25}$$

and they are annihilation and creation operators respectively. By applying the incoming and outgoing creation operators respectively on the vacuums Ω_o and Ψ_o (see def. (4.21) and (4.22)), the *Hilbert spaces* $\mathcal{H}_{in}$ and $\mathcal{H}_{out}$ are constructed. The incoming and outgoing field operators are related by

$$a_{out}[\tilde{f}] = \hat{\phi}[f] = \phi[\hat{f}] = a_{in}[(U\tilde{f})_P] + a^{\dagger}_{in}[(U\tilde{f})_N], \tag{5.26}$$

if $\tilde{f} \in \mathcal{S}_0(\mathbb{R})$, and the *Bogoliubov transformations* are thus given by

$$\begin{aligned} a_{out}[\tilde{f}] &= a_{in}[A\tilde{f}] + a^{\dagger}_{in}[B\tilde{f}], \\ a^{\dagger}_{out}[\tilde{f}] &= a_{in}[B^*\tilde{f}] + a^{\dagger}_{in}[A^*\tilde{f}]. \end{aligned} \tag{5.27}$$

Since $\phi[h] = a_{in}[\tilde{h}]$ if $\tilde{h} \in \mathcal{S}(\mathbb{R}_+)$, we deduce from eq. (5.21) that the field operator commutators are

$$\left[\, a_{in}[\tilde{h}_1],\, a_{in}[\tilde{h}_2]^{\dagger} \,\right] = \langle\, \tilde{h}_2, \tilde{h}_1 \,\rangle, \tag{5.28}$$

$$\left[\, a_{in}[\tilde{h}_1],\, a_{in}[\tilde{h}_2] \,\right] = \left[\, a_{in}[\tilde{h}_1]^{\dagger},\, a_{in}[\tilde{h}_2]^{\dagger} \,\right] = 0, \tag{5.29}$$

where $\tilde{h}_1, \tilde{h}_2 \in \mathcal{S}(\mathbb{R}_+)$. From the invariance of the field commutator (5.23), it is clear that the field operator commutators are also invariant:

$$\left[\, a_{in}[\tilde{h}_1],\, a_{in}[\tilde{h}_2]^{\dagger} \,\right] = \left[\, a_{out}[\tilde{h}_1],\, a_{out}[\tilde{h}_2]^{\dagger} \,\right], \tag{5.30}$$

$$\left[\, a_{in}[\tilde{h}_1],\, a_{in}[\tilde{h}_2] \,\right] = \left[\, a_{out}[\tilde{h}_1],\, a_{out}[\tilde{h}_2] \,\right], \tag{5.31}$$

where $\tilde{h}_1, \tilde{h}_2 \in \mathcal{S}(\mathbb{R}_+)$. Note that eqs (5.28) to (5.31) also imply the fundamental relations (5.17) and (5.18).

The field operator modes $a_{out}(p)$ and $a_{in}(k)$ are defined by

$$a_{out}[\tilde{f}] = \int_0^{\infty} \frac{dp}{2p}\, a_{out}(p)\, \tilde{f}(p), \qquad a_{in}[\tilde{h}] = \int_0^{\infty} \frac{dk}{2k}\, a_{in}(k)\, \tilde{h}(k), \tag{5.32}$$

where $\tilde{h}, \tilde{f} \in \mathcal{S}(\mathbb{R}_+)$. Expansions (5.24) and (5.25) are rewritten as

$$\phi(x) = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} \frac{dk}{2k} \left[a_{in}(k)\, e^{-ikx} + a^{\dagger}_{in}(k)\, e^{ikx} \right], \tag{5.33}$$

$$\hat{\phi}(y) = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} \frac{dp}{2p} \left[a_{out}(p)\, e^{-ipy} + a^{\dagger}_{out}(p)\, e^{ipy} \right]. \tag{5.34}$$

These are representations of the fields $\phi(x)$ and $\hat{\phi}(y)$ in the Hilbert spaces $\mathcal{H}_{in}$ and $\mathcal{H}_{out}$ respectively. The representation of the outgoing field $\hat{\phi}(y)$ in the incoming Hilbert space $\mathcal{H}_{in}$ is deduced from eqs (5.19) and (5.33):

$$\hat{\phi}(y) = \frac{1}{\sqrt{2\pi}} \int_0^\infty \frac{dk}{2k} \left[a_{in}(k)\, e^{-ikx(y)} + a_{in}^\dagger(k)\, e^{ikx(y)} \right]. \tag{5.35}$$

The operator V is defined by the kernel

$$V(k,p) = \frac{1}{2\pi} \int_I dx\, e^{-ikx}\, e^{ipy(x)}, \tag{5.36}$$

where $I = \{\, x(y) \mid y \in \mathbb{R} \,\}$. The operators U and V satisfy the properties

$$V(k,p) = \frac{p}{k}\, U(k,p), \qquad \forall\, k,\, p \in \mathbb{R}, \tag{5.37}$$

$$V^\dagger U = E, \tag{5.38}$$

$$U\, V^\dagger = E \iff I = \mathbb{R}, \tag{5.39}$$

where E is the identity operator. Thus U is non-singular if and only if $I = \mathbb{R}$, and if $I = \mathbb{R}$, we have $U^{-1} = V^\dagger$.

Using the kernel (5.36), the Bogoliubov transformations (5.27) may be rewritten in the form

$$\begin{pmatrix} a_{out}(p) \\ a_{out}^\dagger(p) \end{pmatrix} = \int_0^\infty dk \begin{pmatrix} V(k,p) & -V(-k,p) \\ -V(k,-p) & V(-k,-p) \end{pmatrix} \begin{pmatrix} a_{in}(k) \\ a_{in}^\dagger(k) \end{pmatrix}. \tag{5.40}$$

Equation (5.39) implies that the Bogoliubov transformations (5.27) and (5.40) are invertible if and only if $I = \mathbb{R}$.

Assuming now that $I = \mathbb{R}$, we split the outgoing positive and negative momentum contributions of $V^\dagger \tilde{h}$, defining the operators C and D by

$$(C\tilde{h})(p) = (V^\dagger \tilde{h})_P(p), \qquad (D\tilde{h})(p) = (V^\dagger \tilde{h})_N(p). \tag{5.41}$$

The inverse of the Bogoliubov transformation (5.27) is then given by

$$\begin{aligned} a_{in}[\,\tilde{h}\,] &= a_{out}[\,C\tilde{h}\,] + a_{out}^\dagger[\,D\tilde{h}\,], \\ a_{in}^\dagger[\,\tilde{h}\,] &= a_{out}[\,D^*\tilde{h}\,] + a_{out}^\dagger[\,C^*\tilde{h}\,], \end{aligned} \tag{5.42}$$

if $\tilde{h} \in \mathcal{S}(\mathbb{R}_+)$, or by

$$\begin{pmatrix} a_{in}(k) \\ a_{in}^\dagger(k) \end{pmatrix} = \int_0^\infty dp \begin{pmatrix} U(-k,-p) & -U(-k,p) \\ -U(k,-p) & U(k,p) \end{pmatrix} \begin{pmatrix} a_{out}(p) \\ a_{out}^\dagger(p) \end{pmatrix}. \tag{5.43}$$

The previous results are easily generalized to the *complex* scalar field, for which the field operators $a_{in,out}$ and $b^{\dagger}_{in,out}$ are defined by

$$\phi[h] = a_{in}[\tilde{h}_P] + b^{\dagger}_{in}[\tilde{h}_N], \tag{5.44}$$

$$\hat{\phi}[f] = a_{out}[\tilde{f}_P] + b^{\dagger}_{out}[\tilde{f}_N], \tag{5.45}$$

if $\tilde{h}, \tilde{f} \in \mathcal{S}_0(\mathbb{R})$. The representation of the complex scalar field $\hat{\phi}(y)$ in the Hilbert space $\mathcal{H}_{in}$ is given by

$$\hat{\phi}(y) = \frac{1}{\sqrt{2\pi}} \int_0^\infty \frac{dk}{2k} \left[a_{in}(k)\, e^{-ikx(y)} + b^{\dagger}_{in}(k)\, e^{ikx(y)} \right], \tag{5.46}$$

and the relations

$$\begin{aligned} a_{out}[\tilde{f}] &= a_{in}[A\tilde{f}] + b^{\dagger}_{in}[B\tilde{f}], \\ b^{\dagger}_{out}[\tilde{f}] &= a_{in}[B^*\tilde{f}] + b^{\dagger}_{in}[A^*\tilde{f}], \end{aligned} \tag{5.47}$$

where $\tilde{f} \in \mathcal{S}(\mathbb{R}_+)$, are the associated Bogoliubov transformations.

The outgoing test functions $\tilde{f}_{p_o}$ of mode p_o are defined formally as

$$\tilde{f}_{p_o}(p) = 2p\, \delta(p - p_o). \tag{5.48}$$

This definition is correct only if $p_o > 0$. The null mode $\tilde{f}_{p_o=0}$ is defined as the limit $n \to \infty$ of the series [9]

$$\tilde{f}_0^{(n)}(p) = \frac{\tilde{h}_n(p)}{\langle \tilde{h}, \tilde{h}_n \rangle}, \tag{5.49}$$

where $\tilde{h}(p) = e^{-p^2}$ and $\tilde{h}_n(p) = \tilde{\chi}(np)\, \tilde{h}(p)$, with the function $\tilde{\chi}$ defined by

$$0 \le \tilde{\chi}(p) \le 1, \quad \forall p \in \mathbb{R}, \quad \text{and} \quad \tilde{\chi}(p) = \begin{cases} 0, & \text{if } p \le 0, \\ 1, & \text{if } p \ge 1. \end{cases} \tag{5.50}$$

The series (5.49) satisfies

$$\lim_{n\to\infty} \langle \tilde{f}_0^{(n)}, \tilde{f} \rangle = \tilde{f}(0), \qquad \text{if } \tilde{f} \in \mathcal{S}(\mathbb{R}), \tag{5.51}$$

$$\lim_{n\to\infty} \| \tilde{f}_0^{(n)} \| = 0. \tag{5.52}$$

The generalized functions f_{p_o} ($p_o \ge 0$) are not normalizable and thus they are not associated to a state in the Hilbert space $\mathcal{H}_{out}$.

Let $\{\tilde{f}_i\}_{i=1}^n \subset \mathcal{S}(\mathbb{R}_+)$ be a set of normalized particle test functions. The *n-particle test function* $f^{(n)}$ is defined as

$$f^{(n)} = C\, f_1 \times f_2 \times \ldots \times f_n, \tag{5.53}$$

where $\times$ is the tensor product and C a constant. A product of fields is also defined,

$$\hat{\phi}[f^{(n)}] = C\,\hat{\phi}[f_1]\,\hat{\phi}[f_2]\,...\,\hat{\phi}[f_n], \tag{5.54}$$

and the state denoted $\Psi_{f(n)}$ is given in terms of this product by

$$\Psi_{f(n)} = \hat{\phi}[f^{(n)}]^\dagger\,\Psi_o. \tag{5.55}$$

The state $\Psi_{f(n)}$ is normalized by imposing the equation

$$(\Psi_{f(n)}, \Psi_{f(n)}) = (\Psi_o, \hat{\phi}[f^{(n)}]\,\hat{\phi}[f^{(n)}]^\dagger\,\Psi_o) = 1, \tag{5.56}$$

which fixes the constant C.

6 Observables in the outgoing coordinates

In this section mean values of observables, built into the outgoing coordinates, are computed in the incoming vacuum. These quantities describe the properties of the outgoing particles created by the space-time curvature. The two-point function, energy-momentum tensor, current for the complex scalar field and the mean number of spontaneously created particles for a given outgoing test function are considered. The total mean number of particles is computed and the implementability of U is also considered.

The *outgoing two-point function* $\widehat{W}_o(y,y')$ is defined as the mean value of outgoing fields in the incoming vacuum:

$$\widehat{W}_o(y,y') = (\Omega_o, \hat{\phi}(y)\,\hat{\phi}(y')^\dagger\,\Omega_o). \tag{6.1}$$

This is given from eq. (3.17) by

$$\widehat{W}_o(y,y') = W_o(x(y), x(y')) = -\frac{1}{4\pi}\,\log\left[\,x(y') - x(y) + \mathrm{i}0^+\,\right]. \tag{6.2}$$

The energy-momentum observables in the incoming and outgoing coordinates are given by the products of derivatives of the field at the same point:

$$\Theta(x) = \partial_x\phi(x)^\dagger\,\partial_x\phi(x), \tag{6.3}$$

$$\widehat{\Theta}(y) = \partial_y\hat{\phi}(y)^\dagger\,\partial_y\hat{\phi}(y). \tag{6.4}$$

Their mean value in a given state must thus be regularized. This regularization may be carried out in a covariant way along a geodesic by subtracting the mean value in Minkowski space-time [14], or by ordering the fields normally following a covariant procedure [15]. These two methods must give identical results and their application is made simpler in (asymptotically) flat space-time regions[10]. The regularized mean value of $\widehat{\Theta}(y)$ in the incoming vacuum

[10] The normal order regularization was applied for the Dirac field in asymptotically flat space-time regions by Th. Gallay and G. Wanders [16].

will be computed here in the outgoing (asymptotically) flat space-time region M_F. This is called the *energy-momentum tensor* and will be denoted by $\widehat{T}_o(y)$.

The observables $\Theta_\varepsilon(x)$ and $\widehat{\Theta}_\varepsilon(y)$ are defined by

$$\Theta_\varepsilon(x) \;=\; \frac{1}{2}\left[\partial_x\phi(x)^\dagger\,\partial_x\phi(x+\varepsilon)+\partial_x\phi(x+\varepsilon)^\dagger\,\partial_x\phi(x)\right], \tag{6.5}$$

$$\widehat{\Theta}_\varepsilon(y) \;=\; \frac{1}{2}\left[\partial_y\hat{\phi}(y)^\dagger\,\partial_y\hat{\phi}(y+\varepsilon)+\partial_y\hat{\phi}(y+\varepsilon)^\dagger\,\partial_y\hat{\phi}(y)\right]. \tag{6.6}$$

The energy-momentum tensor $\widehat{T}_o(y)$ regularized by subtraction is given by the limit

$$\widehat{T}_o(y) \;=\; \lim_{\varepsilon\to 0}\,(\Omega_o,\left[\widehat{\Theta}_\varepsilon(y)\,-\,\Theta_\varepsilon(x(y))\right]\Omega_o), \tag{6.7}$$

which is well defined. It is computed using the representation (5.35) or (5.46) of the outgoing field in the incoming Hilbert space $\mathcal{H}_{in}$ and is given by (see appendix A.3)

$$\widehat{T}_o(y) \;=\; -\frac{1}{24\pi}\,\mathrm{S}_y[\,x(y)\,], \tag{6.8}$$

where $\mathrm{S}_y[\,x(y)\,]$ is the Schwartzian derivative of $x(y)$ with respect to y [11]:

$$\mathrm{S}_y[\,x\,] = \left(\frac{x''}{x'}\right)' - \frac{1}{2}\left(\frac{x''}{x'}\right)^2. \tag{6.9}$$

The energy-momentum tensor may also be regularized normally as follows

$$\widehat{T}_o(y) \;=\; (\Omega_o, :\widehat{\Theta}\,(y)\,:_{out}\;\Omega_o), \tag{6.10}$$

where the outgoing normal ordering has to be carried out before computing the incoming vacuum mean value. This definition also implies the result (6.8) but in this case the computation is laborious (see appendix A.4).

From eq. (6.8) the transformation law for the energy-momentum tensor is deduced under the change of coordinates $y = y(z)$

$$\widehat{T}_o(y) \longrightarrow \widehat{\widehat{T}}_o(z) \;=\; y'(z)^2\,\widehat{T}_o(y(z)) - \frac{1}{24\pi}\,\mathrm{S}_z[\,y(z)\,], \tag{6.11}$$

where $\widehat{T}_o(y)$ and $\widehat{\widehat{T}}_o(z)$ are the regularized mean values of the energy-momentum observables in the incoming vacuum in the coordinates y and z respectively.

For the complex scalar field the incoming and outgoing current observables are given by

$$\Upsilon(x) \;=\; i\,\phi(x)^\dagger\,\overleftrightarrow{\partial}_x\,\phi(x), \tag{6.12}$$

$$\widehat{\Upsilon}(y) \;=\; i\,\hat{\phi}(y)^\dagger\,\overleftrightarrow{\partial}_y\,\hat{\phi}(y), \tag{6.13}$$

[11]We have also $\mathrm{S}_y[\,x\,] = \frac{x'''}{x'} - \frac{3}{2}\left(\frac{x''}{x'}\right)^2 = -2\sqrt{x'}\,\partial_y^2\frac{1}{\sqrt{x'}} = \partial_y^2\log x' - \frac{1}{2}(\partial_y\log x')^2$.

and the observables $\Upsilon_\varepsilon(x)$ and $\hat{\Upsilon}_\varepsilon(y)$ are defined as

$$\Upsilon_\varepsilon(x) = i \left[\phi(x+\varepsilon)^\dagger \partial_x \phi(x) - \partial_x \phi(x)^\dagger . \phi(x+\varepsilon) \right], \tag{6.14}$$

$$\hat{\Upsilon}_\varepsilon(y) = i \left[\hat{\phi}(y+\varepsilon)^\dagger \partial_y \hat{\phi}(y) - \partial_y \hat{\phi}(y)^\dagger . \hat{\phi}(y+\varepsilon) \right]. \tag{6.15}$$

The *outgoing current* $\hat{J}_o(y)$ is defined in the subtraction regularization scheme as

$$\hat{J}_o(y) = \lim_{\varepsilon \to 0} (\Omega_o, \left[\hat{\Upsilon}_\varepsilon(y) - \Upsilon_\varepsilon(x(y)) \right] \Omega_o). \tag{6.16}$$

This limit is well defined and is computed in appendix A.5[12]:

$$\hat{J}_o(y) = 0. \tag{6.17}$$

The outgoing current vanishes for any transformation of coordinates $x = x(y)$, i.e. particles and antiparticles are always created locally in pairs.

The outgoing current $\hat{J}_o(y)$ in the normal order regularization scheme is defined by the equation

$$\hat{J}_o(y) = (\Omega_o, : \hat{\Upsilon}(y) :_{out} \Omega_o), \tag{6.18}$$

which also implies the result (6.17) (see appendix A.6).

In the real scalar fields, the *mean number of spontaneously created particles* for a normalized particle test function $\tilde{f} \in \mathcal{S}(\mathbb{R}_+)$ is defined by

$$\bar{N}_o[f] = (\Omega_o, a_{out}[\tilde{f}]^\dagger a_{out}[\tilde{f}] \Omega_o), \tag{6.19}$$

and using the Bogoliubov transformations (5.27) this implies

$$\bar{N}_o[f] = (\Omega_o, a_{in}[B^* \tilde{f}^*] a_{in}^\dagger [B\tilde{f}] \Omega_o). \tag{6.20}$$

This quantity is thus expressed in terms of the Fourier transform $\tilde{f}(p)$ by

$$\bar{N}_o[f] = \| B\tilde{f} \|^2, \tag{6.21}$$

showing that the mean number $\bar{N}_o[f]$ depends only on the negative momentum contributions of the incoming test function $\hat{f}(x)$. $\bar{N}_o[f]$ is also expressed directly in terms of the outgoing test function $f(y)$ using eq. (5.14)

$$\bar{N}_o[f] = -\frac{1}{4\pi} \int_{-\infty}^{+\infty} dy \int_{-\infty}^{+\infty} dy' \, f(y) \log \left[\frac{x(y) - x(y')}{y - y'} \right] f(y')^*. \tag{6.22}$$

It may be checked that the l.h.s. of eq. (6.22) is always positive if $f(y)$ is a particle test function. The results (6.21) and (6.22) are extended to any wave function $\tilde{f} \in L^2(\frac{dp}{2p}, \mathbb{R}_+)$ if $f(y)$ exists a.e. and is integrable.

[12]The same result was obtained for the Dirac field [16].

The mean number of spontaneously created particles *in the mode* f_p, given by eqs (5.48) and (5.49), is defined formally as

$$\bar{N}_o[f_p] = 4p^2 \int_0^\infty \frac{dk}{2k} \mid B(k,p) \mid^2 , \tag{6.23}$$

in agreement with eq. (6.21). The *total mean number* $\bar{N}_o^{tot}$ of spontaneously created particles is defined as the sum of the contributions (6.23) for each mode f_p,

$$\bar{N}_o^{tot} = \int_0^\infty \frac{dp}{2p} \bar{N}_o[f_p] = \int_0^\infty dp\, 2p \int_0^\infty \frac{dk}{2k} \mid B(k,p) \mid^2 , \tag{6.24}$$

which can also be expressed as (see appendix A.7)[13]

$$\bar{N}_o^{tot} = \frac{1}{4\,\pi^2} \int_{-\infty}^{+\infty} dy \int_{-\infty}^{+\infty} dy'\, P \frac{1}{y-y'} \left[\frac{x'(y)}{x(y)-x(y')} - \frac{1}{y-y'} \right] . \tag{6.25}$$

The operator U is said to be *unitarily implementable* if there exists a unitary operator $\mathcal{U} : \mathcal{H}_{in} \to \mathcal{H}_{out}$ which satisfies

$$\phi[f] = \mathcal{U}^\dagger \,\hat{\phi}[f]\, \mathcal{U}, \qquad \forall \tilde{f} \in \mathcal{S}_0(\mathbb{R}). \tag{6.26}$$

If the operator $\mathcal{U}$ exists, the fields ϕ and $\hat{\phi}$ are equivalent representations of the commutator (5.21), in the Hilbert spaces $\mathcal{H}_{in}$ and $\mathcal{H}_{out}$ respectively, and the incoming and outgoing vacuums are related by

$$\Psi_o = \mathcal{U}\, \Omega_o. \tag{6.27}$$

It has been proved that the operator U is unitarily implementable if and only if $\bar{N}_o^{tot}$ is finite [17].

The definition (6.19) of the mean number of spontaneously created particles is generalized to an n-particle normalized test function $f^{(n)}$ by the equation

$$\bar{N}_o[f^{(n)}] = (\Omega_o, N_{out}[f^{(n)}]\, \Omega_o), \tag{6.28}$$

where

$$N_{out}[f^{(n)}] = \hat{\phi}[f^{(n)}]^\dagger \,\hat{\phi}[f^{(n)}]. \tag{6.29}$$

Assuming that the one-particle test functions f_i are orthonormalized

$$\langle \tilde{f}_i, \tilde{f}_j \rangle = \delta_{ij}, \tag{6.30}$$

eq. (6.28) gives

$$N_{out}[f^{(n)}] = N_{out}[f_1]\, N_{out}[f_2] \dots N_{out}[f_n], \tag{6.31}$$

[13]Note that the kernel in the double integral (6.25) is not symmetric as in the case of the Dirac field [16].

where the set of operators $N_{out}[f_i]$ satisfies

$$[\; N_{out}\,[f_i],\; N_{out}\,[f_j]\;] \;=\; 0, \qquad i,j = 1,2,...,n, \tag{6.32}$$

and where $f^{(n)}$ is defined by eqs (5.53) and (5.56). Under the assumption (6.30) it is possible to give a compact formula for the mean number $\bar{N}_o[f^{(n)}]$ using the definitions

$$(f^{(n)} \times f^{(n)*})_S\,(y_1,...,y_{2n}) = \frac{C^2}{(2n)!} \sum_{\tau\in\mathcal{P}_{2n}} f_1(y_{\tau(1)}) \cdots f_n(y_{\tau(n)})\, f_1(y_{\tau(n+1)})^* \cdots f_n(y_{\tau(2n)})^* \tag{6.33}$$

and $G^n = \overbrace{G \times G \times ... \times G}^{n \text{ times}}$, where G is defined by eq. (5.12). This formula is displayed in the following theorem, proved in appendix A.8.

Theorem 2 *If $\{\tilde{f}_i\}_{i=1}^n \subset L^2(\frac{dp}{2p}, \mathbb{R}_+)$ is an orthonormal set of functions such that f_i exists and is integrable ($i = 1,2,...,n$), then*

$$\bar{N}_o[f^{(n)}] \;=\; C^2\,\frac{(2n)!}{2^n\, n!}\, G^n\,[\,(f^{(n)} \times f^{(n)*})_S\,], \tag{6.34}$$

where $f^{(n)}$ is defined by eqs (5.53) and (5.56). Equation (6.34) contains at most $\frac{2^n n!}{(2n)!}$ distinct terms.

7 Scalar field theory in a thermal bath

In this section the one-dimensional massless scalar field is considered in a thermal bath of temperature β^{-1} for null chemical potential. I will restrict the scalar field to the finite interval $[-L, L]$ and impose periodic boundary conditions before taking the "thermodynamic" limit $L \to \infty$. This procedure is necessary to define thermal mean values correctly. The space-time and energy-momentum variables will be denoted here by τ and ω.

The real scalar field $\varphi_L(\tau)$ in the interval $[-L, L]$ is given by

$$\varphi_L(\tau) \;=\; \frac{1}{\sqrt{2L}} \sum_{n=1}^{\infty} \frac{1}{2\omega_n}\left[a_n\, e^{-i\omega_n\tau} + a_n^\dagger\, e^{i\omega_n\tau}\right] + \frac{a_o}{\sqrt{2L}}, \tag{7.1}$$

where $\tau \in [-L, L]$, $\omega_n = n\pi/L$, $a_o \in \mathbb{R}$ and $a_n \in \mathbb{C}$ if $n \in \mathbb{N}$. It is quantized by imposing the field operator commutators

$$\left[a_n,\, a_m^\dagger\right] \;=\; 2\,\omega_n\,\delta_{n,m}, \qquad [\,a_n,\, a_m\,] \;=\; 0, \tag{7.2}$$

where $n, m \geq 1$. These act on a Hilbert space $\mathcal{H}_L$ whose vacuum will be denoted by Φ_o. Equations (7.1) and (7.2) show that the field commutator is given by[14]

$$[\,\varphi_L(\tau), \varphi_L(\tau')\,] \;=\; \frac{i}{4}\,\mathrm{sgn}\,(\tau' - \tau)\left(1 - \frac{|\tau' - \tau|}{L}\right), \tag{7.3}$$

[14] Using the formula $\sum_{n=1}^{\infty} \frac{1}{n}\, \sin(a\,n) = \mathrm{sgn}\,(a)\, \frac{\pi - |a|}{2}$ ($|a| < 2\pi$), with $a = \pi\,(\tau' - \tau)/L$ [18].

where $\tau, \tau' \in [-L, L]$.

The correspondence between field theories for finite and infinite intervals is given by

$$\begin{aligned} R_L, \mathcal{H}_L, \varphi_L &\longleftrightarrow R_\infty, \mathcal{H}, \varphi, \\ \omega_n,\ n \in \mathbb{N} &\longleftrightarrow \omega \in \mathbb{R}_+^*, \\ L\,\delta_{n,m} &\longleftrightarrow \pi\,\delta(\omega - \omega'), \\ \sqrt{\pi}\,a_n &\longleftrightarrow \sqrt{L}\,a(\omega). \end{aligned} \tag{7.4}$$

In the thermodynamic limit the null momentum mode a_o disappears in eq. (7.1) and the commutator (7.3) is then formally equal to eq. (3.24).

The *thermal mean value* of a given observable A is defined by the limit

$$\langle A \rangle_\beta^{Th} = \lim_{L\to\infty} \frac{\mathrm{Tr}_L\left[e^{-\beta H_L}\,A\right]}{\mathrm{Tr}_L\left[e^{-\beta H_L}\right]}, \tag{7.5}$$

where Tr_L is the trace on the Hilbert space $\mathcal{H}_L$ and H_L is the free Hamiltonian given by

$$H_L = \sum_{n=1}^{\infty} \omega_n\, a_n^\dagger\, a_n. \tag{7.6}$$

The partition function $Z_L = \mathrm{Tr}_L\left[e^{-\beta H_L}\right]$ is IR divergent in the thermodynamic limit. Note that the thermal mean value (7.5) is generally well defined although this limit will not necessarily converge to a finite value for any observable A.

From def. (7.5), thermal mean values of field operators are given by

$$\langle a^\dagger(\omega)\,a(\omega')\rangle_\beta^{Th} = \frac{2\omega}{e^{\beta\omega}-1}\,\delta(\omega-\omega'), \tag{7.7}$$

$$\langle a(\omega)\,a^\dagger(\omega')\rangle_\beta^{Th} = \frac{2\omega}{1-e^{-\beta\omega}}\,\delta(\omega-\omega'), \tag{7.8}$$

$$\langle a(\omega)\,a(\omega')\rangle_\beta^{Th} = \langle a^\dagger(\omega)\,a^\dagger(\omega')\rangle_\beta^{Th} = 0, \tag{7.9}$$

where $\omega, \omega' > 0$. Equation (7.7) is proved in appendix A.9. We also have

$$\begin{aligned} &\langle a^\dagger(\omega_n)\ldots a^\dagger(\omega_1)\,a(\omega_1')\ \ldots\ a(\omega_n')\rangle_\beta^{Th} \\ &= \frac{2\omega_1}{e^{\beta\omega_1}-1}\cdots\frac{2\omega_n}{e^{\beta\omega_n}-1}\sum_{\sigma\in\mathcal{P}_n}\delta(\omega_1-\omega'_{\sigma(1)})\ \ldots\ \delta(\omega_n-\omega'_{\sigma(n)}), \end{aligned} \tag{7.10}$$

where $\omega_i, \omega_i' > 0$, $i = 1, 2, \ldots, n$ (see appendix A.10). More generally, the Wick theorem is satisfied for thermal mean values of field operators.

The *thermal two-point function* is defined by

$$W_\beta^{Th}(\tau,\tau') = \langle \phi(\tau)\,\phi(\tau')^\dagger \rangle_\beta^{Th}, \tag{7.11}$$

and satisfies the properties

$$W_\beta^{Th}(\tau,\tau') = W_\beta^{Th}(\tau,\tau'+in\beta), \qquad \forall n \in \mathbb{Z}, \tag{7.12}$$

$$\mathrm{Re}\, W_\beta^{Th}(\tau,\tau') = \sum_{n=-\infty}^{+\infty} \mathrm{Re}\, W_\infty^{Th}(\tau+in\beta,\tau'), \tag{7.13}$$

$\forall \tau,\tau' \in \mathbb{R}$ (see appendix A.11). Using the formula [18, (89.10.4)]

$$(\tau'-\tau)^2 \prod_{n=1}^{\infty} \frac{(\tau'-\tau+in\beta)^2(\tau'-\tau-in\beta)^2}{\beta^4 n^4} = \frac{\beta^2}{\pi^2}\sinh^2\left[\frac{\pi}{\beta}(\tau'-\tau)\right], \tag{7.14}$$

we obtain from eqs (3.17) and (7.13)

$$\mathrm{Re}\, W_\beta^{Th}(\tau,\tau') = -\frac{1}{4\pi}\log\left[\frac{\beta}{\pi}\sinh\left(\frac{\pi}{\beta}\,|\tau'-\tau|\right)\right] + C, \tag{7.15}$$

where C is an infinite constant, hence the thermal mean value (7.11) is infinite.

The thermal two-point function $W_\beta^{Th}(\tau,\tau')$ will thus be redefined as the kernel of a distribution on $\mathcal{S}_0(\mathbb{R}) \times \mathcal{S}_0(\mathbb{R})$ by

$$W_\beta^{Th}[\,f_1 \times f_2^*\,] = \langle \varphi[f_1]\,\varphi[f_2]^\dagger \rangle_\beta^{Th}. \tag{7.16}$$

From eqs (7.1) and (7.7) to (7.9) we obtain

$$W_\beta^{Th}[\,f_1 \times f_2^*\,] = \int_{-\infty}^{+\infty} \frac{d\omega}{2\omega}\,\frac{\tilde{f}_2(\omega)^*\,\tilde{f}_1(\omega)}{1-e^{-\beta\omega}}, \tag{7.17}$$

where $\tilde{f}_1, \tilde{f}_2 \in \mathcal{S}_0(\mathbb{R})$. The following theorem, proved in appendix A.12, gives the correct expression for the kernel $W_\beta^{Th}(\tau,\tau')$.

Theorem 3 *Between kernels of distributions on* $\mathcal{S}_0(\mathbb{R}) \times \mathcal{S}_0(\mathbb{R})$,

$$W_\beta^{Th}(\tau,\tau') = -\frac{1}{4\pi}\log\left\{\frac{\beta}{\pi}\sinh\left[\frac{\pi}{\beta}\left(\tau'-\tau+i0^+\right)\right]\right\}. \tag{7.18}$$

The periodicity property (7.12) is satisfied by (7.18) up to an irrelevant constant.

The *thermal energy-momentum tensor* $T_\beta^{Th}(\tau)$ is defined by the limit

$$T_\beta^{Th}(\tau) = \lim_{\varepsilon\to 0}\left[\langle \Theta_\varepsilon(\tau)\rangle_\beta^{Th} - (\Phi_o, \Theta_\varepsilon(\tau)\,\Phi_o)\right], \tag{7.19}$$

where the observable $\Theta_\varepsilon(\tau)$ is given by

$$\Theta_\varepsilon(\tau) = \frac{1}{2}\left[\partial_\tau\varphi(\tau)^\dagger\,\partial_\tau\varphi(\tau+\varepsilon) + \partial_\tau\varphi(\tau+\varepsilon)^\dagger\,\partial_\tau\varphi(\tau)\right]. \tag{7.20}$$

Using eq. (7.18) we obtain

$$\langle\, \Theta_\varepsilon(\tau)\,\rangle_\beta^{Th} \;=\; -\frac{\pi}{4\beta^2}\,\frac{1}{\sinh^2(\pi\,\varepsilon/\beta)} \;=\; -\frac{1}{4\pi^2\varepsilon^2}+\frac{\pi}{12\beta^2}+\mathcal{O}(\varepsilon^2), \tag{7.21}$$

from which we deduce that $T_\beta^{Th}(\tau)$ depends only on β:

$$T_\beta^{Th}(\tau) \;=\; T_\beta^{Th} \;=\; \frac{\pi}{12\beta^2}, \qquad \forall\,\tau\in\mathbb{R}. \tag{7.22}$$

The *thermal current* $J_\beta^{Th}(\tau)$ associated with the complex scalar field is defined by

$$J_\beta^{Th}(\tau) \;=\; \lim_{\varepsilon\to 0}\Big[\,\langle\, \Upsilon_\varepsilon(\tau)\,\rangle_\beta^{Th} - (\Phi_o, \Upsilon_\varepsilon(\tau)\,\Phi_o)\,\Big], \tag{7.23}$$

where the observable $\Upsilon_\varepsilon(\tau)$ is given by

$$\Upsilon_\varepsilon(\tau) \;=\; i\left[\varphi(\tau+\varepsilon)^\dagger\,\partial_\tau\phi(\tau) - \partial_\tau\varphi(\tau)^\dagger\,.\varphi(\tau+\varepsilon)\right]. \tag{7.24}$$

The limit (7.23) is well defined and is given by

$$J_\beta^{Th}(\tau) \;=\; 0, \tag{7.25}$$

so there is no net local current.

In the real scalar fields, the *thermal mean value of the number of particles* for a normalized particle test function $f\in\mathcal{S}(\mathbb{R}_+)$ is defined by

$$\bar{N}_\beta^{Th}[f] \;=\; \langle\, a[f]^\dagger\, a[f]\,\rangle_\beta^{Th}, \tag{7.26}$$

and from eq. (7.7) we obtain

$$\bar{N}_\beta^{Th}[f] \;=\; \int_0^\infty \frac{d\omega}{2\omega}\,\frac{|\,\tilde{f}(\omega)\,|^2}{e^{\beta\omega}-1}\,. \tag{7.27}$$

This result is extended to any wave function $\tilde{f}\in L^2(\frac{d\omega}{2\omega},\mathbb{R}_+)$ if $f(\tau)$ exists a.e. and is integrable.

We define furthermore the distribution G_β^{Th} on $L^2(\frac{d\omega}{2\omega},\mathbb{R}_+)\times L^2(\frac{d\omega}{2\omega},\mathbb{R}_+)$ by

$$G_\beta^{Th}(f_1\times f_2^*) \;=\; \int_0^\infty \frac{d\omega}{2\omega}\,\frac{\tilde{f}_2(\omega)^*\,\tilde{f}_1(\omega)}{e^{\beta\omega}-1}\,. \tag{7.28}$$

The following theorem gives an expression for the thermal mean value of the number of particles for an n-particle normalized test function $f^{(n)}$:

$$\bar{N}_\beta^{Th}[f^{(n)}] \;=\; \langle\, \hat{\phi}[f^{(n)}]^\dagger\,\hat{\phi}[f^{(n)}]\,\rangle_{\beta,out}^{Th}. \tag{7.29}$$

It is easily proved using eq. (7.10).

Theorem 4 *If $\{\tilde{f}_i\}_{i=1}^n \subset L^2(\frac{d\omega}{2\omega}, \mathbb{R}_+)$ is an orthonormal set of functions such that f_i exists and is integrable ($i = 1, 2, ..., n$), then*

$$\bar{N}_\beta^{Th}[f^{(n)}] = C^2 \sum_{\sigma\in\mathcal{P}_n} G_\beta^{Th}(f_1 \times f_{\sigma(1)}^*)\, G_\beta^{Th}(f_2 \times f_{\sigma(2)}^*) \ldots G_\beta^{Th}(f_n \times f_{\sigma(n)}^*), \qquad (7.30)$$

where $f^{(n)}$ is defined as in eqs (5.53) and (5.56).

A state $\Phi \in \mathcal{H}$ is said to be a *thermal state* of temperature β^{-1} if it satisfies the equation [10]

$$(\Phi, A_\tau\, B\, \Phi) = (\Phi, B\, A_{\tau+\mathrm{i}\beta}\, \Phi), \qquad (7.31)$$

where A and B are two operators and where we have defined

$$A_\tau = e^{\mathrm{i}\tau H}\, A\, e^{-\mathrm{i}\tau H}, \qquad (7.32)$$

where H is the free Hamiltonian. Equation (7.31) is known as the KMS condition. It can also be written in the equivalent form [12]

$$(\Phi, A\, B\, \Phi) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\omega \int_{-\infty}^{+\infty} d\tau\, \frac{e^{\mathrm{i}\omega\tau}}{1 - e^{-\beta\omega}}\, (\Phi, [\, A_\tau, B\,]\, \Phi). \qquad (7.33)$$

In the particular case where $A = \varphi(\tau)$ and $B = \varphi(\tau')$, we obtain

$$(\Phi, \varphi(\tau)\, \varphi(\tau')\, \Phi) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{d\omega}{2\omega}\, \frac{e^{\mathrm{i}\omega\,(\tau'-\tau)}}{1 - e^{-\beta\omega}} \qquad (7.34)$$

from the commutator (7.3). The integral in the r.h.s. is IR divergent and is formally equal to the kernel of $W_\beta^{Th}[\, f_1 \times f_2^*\,]$ (see eq. (7.17)). The KMS condition is thus restated as an equality between kernels of distributions on $\mathcal{S}_0(\mathbb{R}) \times \mathcal{S}_0(\mathbb{R})$ in the form

$$(\Phi, \varphi(\tau)\, \varphi(\tau')\, \Phi) = W_\beta^{Th}(\tau, \tau'), \qquad (7.35)$$

where $W_\beta^{Th}(\tau, \tau')$ is given by eq. (7.18). If this last equation is satisfied on a interval I for a given state Φ, $\forall\, \tau, \tau' \in I$, we shall say that Φ is a thermal state on this interval.

8 Spontaneous creation of particles

So far the massless scalar field has been studied in 2D curved space-times. In this section the results obtained previously are applied to the relativistic black hole model, for which the transformation of coordinates $x = x(y)$ is given by (2.14)

$$x(y) = \Delta - e^{-My}, \qquad \forall\, y \in \mathbb{R}. \qquad (8.1)$$

The kernel $U(k,p)$, defined by eq. (5.8), can be explicitly computed for this model and is given by (see appendix A.13)

$$U(k,p) \;=\; \frac{e^{-ik\Delta}\, e^{-i\Omega(\frac{p}{M})}\, e^{i\frac{p}{M}\log|k|}}{\sqrt{2\pi}\,M} \left[\frac{\theta(k)}{\sqrt{p(1-e^{-\frac{2\pi}{M}p})}} + \frac{\theta(-k)}{\sqrt{p(e^{\frac{2\pi}{M}p}-1)}} \right], \qquad (8.2)$$

$\forall\, k, p \neq 0$, where $\Omega(p) = \text{Arg}\,[\Gamma(ip)]$. Note that this kernel satisfies the property

$$|\,U(k,p)\,| \;=\; e^{\text{sgn}(k)\pi p}\;|\,U(-k,p)\,|\,. \qquad (8.3)$$

The Bogoliubov transformation (5.40) is obtained from eq. (8.2) and is given by

$$a_{out}(p) \;=\; \sqrt{\frac{Mp}{2\pi}}\; e^{-ik\Delta}\, e^{-i\Omega(\frac{p}{M})} \int_0^\infty \frac{dk}{k}\, e^{i\frac{p}{M}\log k} \left[\frac{a_{in}(k)}{\sqrt{1-e^{-\frac{2\pi}{M}p}}} + \frac{a_{in}^\dagger(k)}{\sqrt{e^{\frac{2\pi}{M}p}-1}} \right], \qquad (8.4)$$

where $p > 0$. The kernel (8.2) and the Bogoliubov transformation (8.4) are not invertible (see discussion following eq. (5.39)).

Equations (6.23) and (8.2) show that the mean number of spontaneously created particles for the mode f_p (5.48) is IR and UV divergent in the incoming momentum k:

$$\bar{N}_o[f_p] \;=\; \frac{1}{\pi M}\, \frac{2p}{e^{\frac{2\pi}{M}p}-1} \int_0^\infty \frac{dk}{2k} \;=\; \infty, \qquad (8.5)$$

if $p > 0$. This result is also true for $p = 0$ in which case f_0 is given by def. (5.49). The total mean number of spontaneously created particles is moreover IR divergent in the outgoing momentum p (see eq. (6.24))

$$\bar{N}_o^{tot} \;=\; \infty, \qquad (8.6)$$

and the operator U is therefore not implementable (see discussion after eq. (6.27)).

In the following, the mean values of outgoing observables in the incoming vacuum are compared with their corresponding thermal mean values in the Hilbert space $\mathcal{H}_{out}$, given by (see eq. (7.5))

$$\langle\, A\,\rangle^{Th}_{\beta,out} \;=\; \lim_{L\to\infty} \frac{\underset{out}{\text{Tr}}_L \left[e^{-\beta H_{L,out}}\, A \right]}{\underset{out}{\text{Tr}}_L \left[e^{-\beta H_{L,out}} \right]}. \qquad (8.7)$$

This enables us to establish the thermal properties of the radiation emitted, and in particular to determine its temperature.

The outgoing two-point function (6.1) is given for the transformation (8.1) by

$$\widehat{W}_o(y,y') \;=\; -\frac{1}{4\pi} \log\left(e^{-My} - e^{-My'} + i0^+ \right). \qquad (8.8)$$

Writing the thermal two-point function (7.18) in the form

$$W^{Th}_{\beta,out}(y,y') = -\frac{1}{4\beta}(y+y') - \frac{1}{4\pi}\log\left[\frac{\beta}{2\pi}\left(e^{-\frac{2\pi}{\beta}y} - e^{-\frac{2\pi}{\beta}y'} + i0^+\right)\right], \qquad (8.9)$$

we deduce that the two-point functions (8.8) and (8.9) are equivalent everywhere as kernels of distributions on $\mathcal{S}_0(\mathbb{R}) \times \mathcal{S}_0(\mathbb{R})$, if and only if $\beta = \frac{2\pi}{M}$:

$$\widehat{W}_o(y,y') = W^{Th}_{\frac{2\pi}{M},out}(y,y'), \qquad \forall y,y' \in \mathbb{R}. \qquad (8.10)$$

We conclude from this last equation that the incoming vacuum Ω_o is a thermal state of temperature $\frac{M}{2\pi}$ in the outgoing coordinates on $\mathbb{R}$.

The energy-momentum tensor is computed from eq. (6.8) and is given by

$$\widehat{T}_o(y) = \frac{M^2}{48\pi}, \qquad \forall y \in \mathbb{R}; \qquad (8.11)$$

hence we deduce from eq. (7.22) that it is thermal

$$\widehat{T}_o(y) = T^{Th}_{\frac{2\pi}{M},out}, \qquad \forall y \in \mathbb{R}, \qquad (8.12)$$

and that the associated temperature is also given by $\frac{M}{2\pi}$ for all $y \in \mathbb{R}$.

We consider now the mean number of spontaneously created particles for a given normalized particle function f. If f is a Schwartz function, eq. (6.22) shows that $\bar{N}_o[f]$ is always finite for the transformation (8.1):

$$\bar{N}_o[f] < \infty, \qquad \forall \tilde{f} \in \mathcal{S}(\mathbb{R}_+). \qquad (8.13)$$

The mean number of particles $\bar{N}_o[f]$ may be explicitly computed from eq. (6.21) and (8.2) and is given by (see appendix A.14)

$$\bar{N}_o[f] = \int_0^\infty \frac{dp}{2p}\,\frac{|\tilde{f}(p)|^2}{e^{\frac{2\pi}{M}p} - 1}. \qquad (8.14)$$

This result shows that $\bar{N}_o[f]$ may also be infinite. For example, defining the test functions $\tilde{f}_\alpha \in L^2(\frac{dp}{2p}, \mathbb{R}_+)$ by

$$\tilde{f}_\alpha(p) = C_\alpha\,\theta(p)\,p^\alpha\,e^{-p^2}, \qquad \alpha > 0, \qquad (8.15)$$

where C_α is a normalization constant, we have the equivalence

$$\bar{N}_o[f_\alpha] = \infty \iff \alpha \le 1/2. \qquad (8.16)$$

If $\alpha \le 1/2$, $\bar{N}_o[f_\alpha]$ is IR divergent in the outgoing momentum p.

Comparing eq. (8.14) with the thermal expression (7.27), we deduce that the mean number of spontaneously created particles is thermal

$$\bar{N}_o[f] = \bar{N}^{Th}_{\frac{2\pi}{M},out}[f], \qquad (8.17)$$

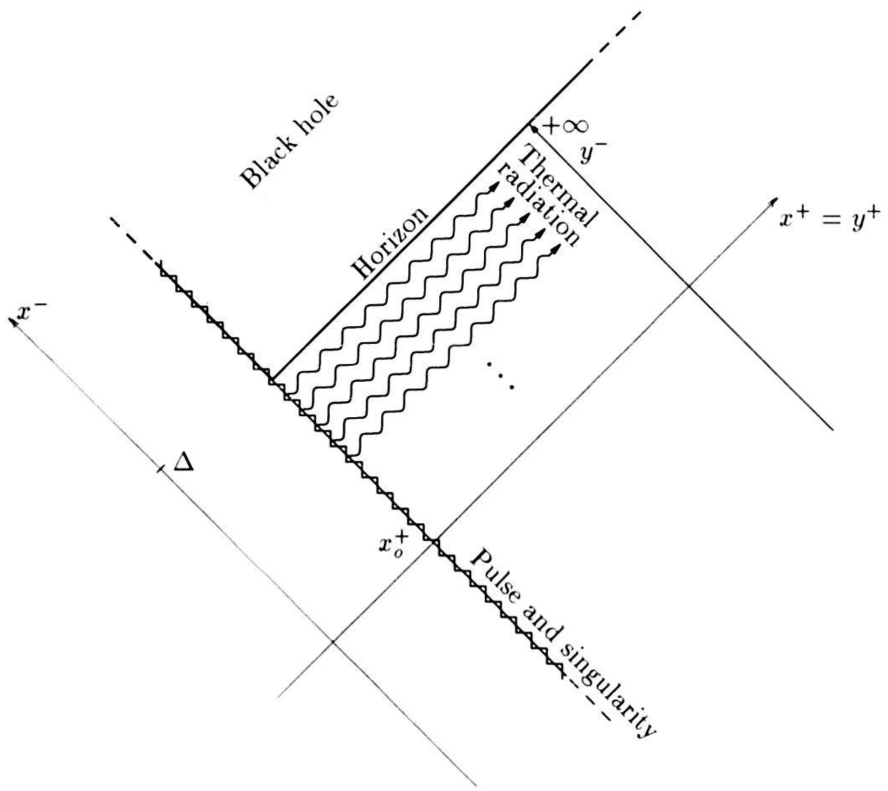

Figure 2: The relativistic black hole and its thermal radiation.

and that the associated temperature is also given by $\frac{M}{2\pi}$. This last result is also true for a normalized n-particle test function $f^{(n)}$. This can easily be proved (see appendix A.15) in the special case for which the functions f_i are orthonormalized, as stated in the following theorem.

Theorem 5 *If $\{\tilde{f}_i\}_{i=1}^n \subset L^2(\frac{dp}{2p}, \mathbb{R}_+)$ is a set of normalized test functions such that f_i exists and is integrable ($i = 1, 2, ..., n$), then*

$$\bar{N}_o[f^{(n)}] = \bar{N}^{Th}_{\frac{2\pi}{M},out}[f^{(n)}], \tag{8.18}$$

where $f^{(n)}$ is defined by eqs (5.53) and (5.56).

9 Conclusions

This new space-time model, based on the "$R = T$" relativistic theory, describes the formation of a black hole whose semi-classical approach is straightforward. This black hole emits an infinity of massless particles in each outgoing momentum mode. The emission is thermal in the sense that mean values in the incoming vacuum of observables constructed in the outgoing coordinates are equal to their thermal averages:

$$(\Omega_o, A\,\Omega_o) = \langle A \rangle^{Th}_{\frac{2\pi}{M},out}. \tag{9.1}$$

Immediately after the formation of the black hole this result is valid everywhere, and not only near the horizon (see figure 2). Equation (9.1) shows that the temperature of the radiation is given by

$$T_{\text{radiation}} = \frac{M}{2\pi}, \tag{9.2}$$

and it is proportional to the relative amplitude of the localized curvature (2.8). The radiation emitted by the black hole is thus described by an outgoing density matrix which is thermal.

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A Appendices

If f is an integrable function, the primitives $F(y)$ and $\hat{F}(x)$ are defined as

$$F(y) = \int_{-\infty}^{y} dy'\, f(y'), \qquad \hat{F}(x) = \int_{x(-\infty)}^{x} dx'\, \hat{f}(x'). \tag{A.1}$$

They are related by

$$\hat{F}(x(y)) = F(y), \qquad \forall y \in \mathbb{R}, \tag{A.2}$$

and satisfy

$$\tilde{F}(p) = \mathrm{i}p\, \tilde{f}(p), \qquad \tilde{\hat{F}}(k) = \mathrm{i}k\, \tilde{\hat{f}}(k), \tag{A.3}$$

$$F(-\infty) = F(+\infty) = \hat{F}(x(-\infty)) = \hat{F}(x(+\infty)) = 0, \tag{A.4}$$

if $\tilde{f}(0) = 0$.

A.1 Proof of eq. (5.14)

Definitions (A.1) show that

$$\langle B\tilde{f}_2, B\tilde{f}_1 \rangle = -\frac{\mathrm{i}}{2}\int_0^{\infty} dk\, \tilde{\hat{f}}_2(-k)^*\, \tilde{\hat{F}}_1(-k) = \frac{1}{4\pi}\int_I dx\, \hat{f}_2(x)^* \int_I dx' \frac{\hat{F}_1(x')}{x' - x + \mathrm{i}0^+}, \tag{A.5}$$

where $I = \{\, x(y) \mid y \in \mathbb{R} \,\}$. Integrating by parts we obtain

$$\frac{1}{4\pi}\int_I dx'\, \frac{\hat{F}_1(x')}{x' - x + \mathrm{i}0^+} = -\frac{1}{4\pi}\int_{-\infty}^{+\infty} dy'\, f_1(y') \log|x(y') - x(y)| - \frac{\mathrm{i}}{4}\hat{F}_1(x). \tag{A.6}$$

The transformations (5.6) and (A.2) imply

$$-\tfrac{\mathrm{i}}{4}\int_I dx\,\hat{f}_2(x)^*\,\hat{F}_1(x) \;=\; -\tfrac{\mathrm{i}}{4}\int_{-\infty}^{+\infty} dy\,f_2(y)^* F_1(y) \;=\; -\tfrac{1}{2}\int_0^{\infty}\frac{dp}{2p}\,\tilde{f}_2(p)^*\,\tilde{f}_1(p), \qquad (A.7)$$

and from eqs (A.5) to (A.7)

$$\begin{aligned}\langle B\tilde{f}_2, B\tilde{f}_1\rangle \;=\; & -\tfrac{1}{4\pi}\int_{-\infty}^{+\infty} dy\int_{-\infty}^{+\infty} dy'\,f_1(y')\,\log|x(y')-x(y)|\;f_2(y)^* \\ & -\tfrac{1}{2}\int_0^{\infty}\frac{dp}{2p}\,\tilde{f}_2(p)^*\,\tilde{f}_1(p).\end{aligned} \qquad (A.8)$$

Restricting eq. (A.8) to the identity transformation $x(y) = y$, we obtain

$$\int_0^{\infty}\frac{dp}{2p}\,\tilde{f}_2(p)^*\,\tilde{f}_1(p) \;=\; -\frac{1}{2\pi}\int_{-\infty}^{+\infty} dy\int_{-\infty}^{+\infty} dy'\,f_1(y')\,\log|y'-y|\;f_2(y)^*, \qquad (A.9)$$

and hence the result (5.14) from eq. (A.8).

A.2 Proof of eq. (5.22)

The definitions (A.1) and transformations (5.6) and (A.2) show that

$$\begin{aligned}\int_{-\infty}^{+\infty}\frac{dk}{2k}\,\hat{\tilde{f}}_2(k)^*\,\hat{\tilde{f}}_1(k) \;&=\; \tfrac{\mathrm{i}}{2}\int_{-\infty}^{+\infty} dk\,\hat{\tilde{f}}_2(k)^*\,\hat{\tilde{F}}_1(k) \;=\; \tfrac{\mathrm{i}}{2}\int_{-\infty}^{+\infty} dx\,\hat{f}_2(x)^*\,\hat{F}_1(x) \\ &=\; \tfrac{\mathrm{i}}{2}\int_{-\infty}^{+\infty} dy\,f_2(y)^*\,F_1(y) \;=\; \int_{-\infty}^{+\infty}\frac{dp}{2p}\,\tilde{f}_2(p)^*\,\tilde{f}_1(p).\end{aligned}$$

A.3 First proof of eq. (6.8)

The energy-momentum tensor is computed here from definition (6.7). From the field representation (5.35) or (5.46) of the field $\hat{\phi}(y)$ in the Hilbert space $\mathcal{H}_{in}$ we deduce

$$(\Omega_o, \partial_y\hat{\phi}(y)^\dagger\,\partial_y\hat{\phi}(y+\varepsilon)\,\Omega_o) \;=\; \frac{x'(y)\,x'(y+\varepsilon)}{4\pi}\int_0^{\infty} dk\;k\;e^{\mathrm{i}k\,[x(y+\varepsilon)-x(y)]}. \qquad (A.10)$$

The formula

$$\int_0^{\infty} dk\,k\;e^{\mathrm{i}kx} \;=\; -\frac{1}{(x+\mathrm{i}0^+)^2}, \qquad (A.11)$$

and eq. (A.10) show that

$$\hat{T}_o(y) \;=\; -\frac{1}{4\pi}\lim_{\varepsilon\to 0}\left\{\frac{x'(y+\varepsilon)\,x'(y)}{[\,x(y+\varepsilon)-x(y)\,]^2} - \frac{1}{\varepsilon^2}\right\}. \qquad (A.12)$$

This limit is well defined and by expanding at $\varepsilon = 0$ we obtain (6.8).

A.4 Second proof of eq. (6.8)

The energy-momentum tensor is computed here from definition (6.10) and for the real scalar field. Ordering normally the field operators and using the equations

$$(\Omega_o, a_{out}(p)\, a_{out}(p')\, \Omega_o) = 4\, p\, p' \int_0^\infty \frac{dk}{2k} A(k,p)\, B(k,p'), \quad (A.13)$$

$$(\Omega_o, a_{out}(p)^\dagger\, a_{out}(p')\, \Omega_o) = 4\, p\, p' \int_0^\infty \frac{dk}{2k} B(k,p)^*\, B(k,p'), \quad (A.14)$$

deduced from the Bogoliubov transformations (5.27), and then integrating on the momentum variables, we obtain

$$(\Omega_o, :\hat{\Theta}(y):_{out}\, \Omega_o) = -\frac{1}{16\pi^3} \int_{-\infty}^{+\infty} dy' \int_{-\infty}^{+\infty} dy'' \frac{1}{[\, x(y') - x(y'') - i0^+\,]^2} \times \quad (A.15)$$

$$\left[\frac{1}{(y-y'+i0^+)(y-y''+i0^+)} + \frac{1}{(y-y'-i0^+)(y-y''-i0^+)} - \frac{2}{(y-y'+i0^+)(y-y''-i0^+)} \right].$$

Using[15]

$$\frac{1}{y \pm i0^+} = P\frac{1}{y} \mp i\pi\, \delta(y), \qquad \frac{1}{(\, x \pm i0^+)^2} = P\frac{1}{x^2} \pm i\pi\, \delta'(x), \quad (A.16)$$

we deduce the result

$$\hat{T}_o(y) = -\frac{1}{4\pi} \int_{-\infty}^{+\infty} dy' \int_{-\infty}^{+\infty} dy''\, \delta(y-y')\, \delta(y-y'') \left\{ \frac{x'(y')\, x'(y'')}{[\, x(y') - x(y'')\,]^2} - \frac{1}{(y'-y'')^2} \right\} \quad (A.17)$$

which again gives the limit (A.12).

A.5 First proof of eq. (6.17)

The outgoing current is computed from def. (6.16). From the field representation (5.46) of the field $\hat{\phi}(y)$ in the Hilbert space $\mathcal{H}_{in}$ we deduce

$$(\Omega_o, \hat{\phi}(y+\varepsilon)^\dagger\, \partial_y \hat{\phi}(y)\, \Omega_o) = \frac{i}{4\pi} \frac{x'(y)}{x(y+\varepsilon) - x(y) - i0^+}, \quad (A.18)$$

and

$$(\Omega_o, \hat{\Upsilon}_\varepsilon(y)\, \Omega_o) = (\Omega_o, \Upsilon_\varepsilon(x(y))\, \Omega_o) = 0, \quad (A.19)$$

from which eq. (6.17) follows.

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[15] We have defined $P\frac{1}{x^m} = \frac{(-1)^{m-1}}{2(m-1)!} \lim_{\varepsilon\to 0} \frac{d^m}{dx^m} \log(x^2+\varepsilon^2)$.

A.6 Second proof of eq. (6.17)

The outgoing current is computed from def. (6.18). Using analogous relations to (A.13) and (A.14) for the complex scalar field, and the equality

$$\int_{-\infty}^{+\infty} \frac{dk}{k} U(k,p)\, U(k,p')^* \;=\; \frac{1}{p}\, \delta\,(p-p'), \tag{A.20}$$

deduced from eq. (5.22), we obtain again eq. (6.17).

A.7 Proof of eq. (6.25)

We follow here ref. [16]. Equations (6.21) and (6.22) imply

$$\int_0^{\infty} \frac{dk}{2k} B(k,p)\, B(k,p')^* \;=\; -\frac{1}{8\pi^2} \int_{-\infty}^{+\infty} dy \int_{-\infty}^{+\infty} dy'\, e^{ipy}\, e^{-ipy'} \log\left[\frac{x(y)-x(y')}{y-y'}\right] \cdot \tag{A.21}$$

Integrating by parts, we deduce from eqs (6.24) and (A.21)

$$\bar{N}_o^{tot} \;=\; \frac{1}{4\pi^2} \int_{-\infty}^{+\infty} dy \int_{-\infty}^{+\infty} dy'\, \frac{1}{y-y'+\mathrm{i}0^+} \left[\frac{x'(y)}{x(y)-x(y')} - \frac{1}{y-y'}\right]. \tag{A.22}$$

The expression in the square brackets is well defined in the limit $y' \to y$:

$$\lim_{y'\to y} \left[\frac{x'(y)}{x(y)-x(y')} - \frac{1}{y-y'}\right] \;=\; \partial_y \log \sqrt{x'(y)} \cdot \tag{A.23}$$

The double-integral (A.22) contains the imaginary contribution $\mathrm{i}\pi\, \delta(y-y')$ whose regularized integral vanishes,

$$\mathrm{i} \log \sqrt{x'(y)}\; e^{-\varepsilon|y|} \Big|_{-\infty}^{+\infty} \;=\; 0, \tag{A.24}$$

where $\varepsilon > 0$. Equation (A.22) then implies the result (6.25).

A.8 Proof of eq. (6.34)

By definition

$$\bar{N}_o[f^{(n)}] \;=\; (\Omega_o, \hat{\phi}[f_n]^\dagger \ldots \hat{\phi}[f_2]^\dagger\, \hat{\phi}[f_1]^\dagger\, \hat{\phi}[f_1]\, \hat{\phi}[f_2] \ldots \hat{\phi}[f_n]\, \Omega_o). \tag{A.25}$$

Defining

$$\mathring{f}_i \;=\; \begin{cases} f_i, & i = 1,2,\ldots,n, \\ f_{i-n}^*, & i = n+1, n+2, \ldots, 2n, \end{cases} \tag{A.26}$$

and assuming $\langle f_i, f_j \rangle = \delta_{ij}$, we deduce from theorem 1 and for the real scalar field:

$$(\Omega_o, \hat{\phi}[\mathring{f}_i]\, \hat{\phi}[\mathring{f}_j]\, \Omega_o) \quad = \quad G(\mathring{f}_i \times \mathring{f}_j), \qquad i,j = 1,2,...,2n. \tag{A.27}$$

Using Wick's theorem, we obtain from eqs (A.25) and (A.27) the result

$$\bar{N}_o[f^{(n)}] \quad = \quad \frac{C^2}{n!\, 2^n} \sum_{\tau \in \mathcal{P}_{2n}} G(\mathring{f}_{\tau(1)} \times \mathring{f}_{\tau(2)}) \ \ldots \ G(\mathring{f}_{\tau(2n-1)} \times \mathring{f}_{\tau(2n)}), \tag{A.28}$$

which is equivalent to eq. (6.34).

A.9 Proof of eq. (7.7)

I follow here ref. [10]. The physical system is restricted to the interval $[-L, L]$. The partition function Z_L is given by

$$Z_L \quad = \quad \sum_{n_1, n_2, \ldots = 0}^{\infty} \exp\left[-\beta \sum_{k=1}^{\infty} n_k \omega_k\right] \quad = \quad \prod_{k=1}^{\infty} \frac{1}{1 - e^{-\beta\omega_k}}, \tag{A.29}$$

and is IR divergent in the limit $L \to \infty$. We have furthermore

$$\begin{aligned} \mathrm{Tr}_L \left[\, e^{-\beta H_L}\, a_i^\dagger\, a_j \,\right] \quad &= \quad \sum_{n_1, n_2, \ldots = 0}^{\infty} \exp\left[-\beta \sum_{k=1}^{\infty} n_k\, \omega_k\right] 2n_i\, \omega_i\, \delta_{i,j} \\ &= \quad -Z_L\, (1 - e^{-\beta\omega_i}) \frac{\partial}{\partial\beta} \sum_{n_i=0}^{\infty} e^{-\beta n_i \omega_i}\, 2\delta_{i,j} \quad = \quad Z_L\, \frac{2\omega_i}{e^{\beta\omega_i} - 1}\, \delta_{i,j}. \end{aligned} \tag{A.30}$$

Equations (A.29) and (A.30) imply the result (7.7) in the thermodynamic limit.

A.10 Proof of eq. (7.10)

We assume for simplicity that $n = 2$. Similar computations to those of appendix A.9 lead to the result

$$Z_L^{-1}\ \mathrm{Tr}_L \left[e^{-\beta H_L}\, a_i^\dagger\, a_j^\dagger\, a_k\, a_l\right] \quad = \quad \frac{4\omega_i\, \omega_j}{(e^{\beta\omega_i} - 1)(e^{\beta\omega_j} - 1)}\, (\,\delta_{i,l}\, \delta_{j,k} + \delta_{i,k}\, \delta_{j,l}\,), \tag{A.31}$$

which is also valid in the particular case $i = j = k = l$. In the thermodynamic limit, eq. (7.10) is then deduced for $n = 2$. Note that a hypothetical supplementary term like $\delta_{i,j,k,l}$ in eq. (A.31) could not survive in the thermodynamic limit.

A.11 Proof of eqs (7.12) and (7.13)

Equation (7.12) is deduced from def. (7.5) using the cyclic property of the trace. To prove eq. (7.13) the physical system is restricted to the interval $[-L, L]$, for which the thermal two-point function (7.11) will be denoted by $W^{Th}_{\beta,L}(t,t')$. Equations (7.1) and (7.2) show that

$$W^{Th}_{\beta,L}(\tau,\tau') = \frac{1}{2L}\sum_{i=1}^{\infty}\frac{1}{2\omega_i}\left[\frac{e^{i\omega_i(\tau-\tau')}}{e^{\beta\omega_i}-1}+\frac{e^{i\omega_i(\tau'-\tau)}}{1-e^{-\beta\omega_i}}\right], \tag{A.32}$$

where we have used the discretized version of eqs (7.7) to (7.9). Noting that

$$\frac{1}{1-e^{-\beta\omega_i}} = \sum_{n=0}^{\infty} e^{-n\beta\omega_i}, \tag{A.33}$$

we obtain

$$W^{Th}_{\beta,L}(\tau,\tau') = \frac{1}{2L}\sum_{i=1}^{\infty}\frac{1}{2\omega_i}\left[\sum_{n=1}^{\infty} e^{i\omega_i(\tau-\tau'+in\beta)}+\sum_{n=0}^{\infty} e^{-i\omega_i(\tau-\tau'-in\beta)}\right], \tag{A.34}$$

from which we deduce

$$\mathrm{Re}\, W^{Th}_{\beta,L}(\tau,\tau') = \sum_{n=-\infty}^{+\infty} \mathrm{Re}\, W^{Th}_{\infty,L}(\tau+in\beta,\tau'). \tag{A.35}$$

We obtain the result (7.13) by taking the thermodynamic limit of this last equation.

A.12 Proof of eq. (7.18)

We define the primitives of $f_i \in \mathcal{S}_0(\mathbb{R})$ as $F_i(t) = \int_{-\infty}^{t} dt'\, f_i(t')$, $i = 1, 2$. Integrating eq. (7.17) twice by parts we obtain

$$W^{Th}_{\beta}[f_1\times f_2^*] = \frac{1}{4\pi}\int_{-\infty}^{+\infty} d\tau\int_{-\infty}^{+\infty} d\tau'\, F_1(\tau)\, F_2(\tau')^*\int_{-\infty}^{+\infty} d\omega\, e^{i\omega(\tau'-\tau)}\,\frac{\omega}{1-e^{-\beta\omega}}\,. \tag{A.36}$$

We interpret τ' as $\tau' + i0^+$ to regularize this integral. Using the formulae [19]

$$2\int_0^{\infty} d\omega\, \cos[\omega(\tau'-\tau)]\,\frac{\omega}{e^{\beta\omega}-1} = \frac{1}{(\tau'-\tau)^2}-\left(\frac{\pi}{\beta}\right)^2\frac{1}{\sinh^2[\beta(\tau'-\tau)/\pi]}, \tag{A.37}$$

$$\int_{-\infty}^{+\infty} d\omega\, \sin[\omega(\tau'-\tau)]\,\frac{\omega}{1-e^{-\beta\omega}} = \int_0^{\infty} d\omega\, \sin[\omega(\tau'-\tau)]\,\omega, \tag{A.38}$$

we deduce from eq. (A.36)

$$\begin{aligned} W^{Th}_{\beta}[f_1\times f_2^*] &= \frac{1}{4\pi}\int_{-\infty}^{+\infty} d\tau\int_{-\infty}^{+\infty} d\tau'\, F_1(\tau)\, F_2(\tau')^*\int_0^{\infty} d\omega\, e^{i\omega(\tau'-\tau)}\,\omega \\ &+ \frac{1}{4\pi}\int_{-\infty}^{+\infty} d\tau\int_{-\infty}^{+\infty} d\tau'\, F_1(\tau)\, F_2(\tau')^*\,\partial_\tau\,\partial_{\tau'}\left\{\log(\tau'-\tau)-\log\sinh\left[\frac{\pi}{\beta}(\tau'-\tau)\right]\right\}. \end{aligned} \tag{A.39}$$

Performing again a double integration by parts we obtain

$$W^{Th}_{\beta}[f_1\times f_2^*] = -\frac{1}{4\pi}\int_{-\infty}^{+\infty} d\tau\int_{-\infty}^{+\infty} d\tau'\, f_1(\tau)\, f_2(\tau')^*\,\log\sinh\left[\frac{\pi}{\beta}(\tau'-\tau)\right]. \tag{A.40}$$

The kernel $W^{Th}_{\beta}(\tau,\tau')$ is contained in this double integral. The arbitrary constant is chosen so as to obtain the expression (3.17) for the two-point function in the limit $\beta\to\infty$.

A.13 Proof of eq. (8.2)

Definition (5.36) and eq. (8.1) imply that the kernel of V is given by

$$V(k,p) \;=\; e^{-\mathrm{i}k\Delta}\, V_o\left(k, \frac{p}{M}\right) \tag{A.41}$$

where we have defined

$$V_o(k,p) \;=\; \frac{1}{2\pi}\int_0^\infty dx\, e^{\mathrm{i}kx}\, x^{-\mathrm{i}p}. \tag{A.42}$$

Changing to the variable $s = |k|\,x$ we obtain

$$V_o(k,p) \;=\; \frac{1}{2\pi}\,\frac{1}{|\,k\,|^{1-\mathrm{i}p}}\,[\,\theta(k)\,J(-p) + \theta(-k)\,J(p)^*\,], \tag{A.43}$$

where we have defined

$$J(p) \;=\; \int_0^\infty ds\, e^{\mathrm{i}s}\, s^{\mathrm{i}p}. \tag{A.44}$$

This integral is computed by deforming the contour along $\mathbb{R}_+$ to the imaginary positive axis:

$$J(p) \;=\; -p\; e^{-\frac{\pi}{2}p}\; \Gamma(\mathrm{i}p). \tag{A.45}$$

Since

$$|\,\Gamma(\mathrm{i}p)\,|^2 \;=\; \frac{\pi}{p\,\sinh(\pi p)}, \tag{A.46}$$

we obtain

$$J(p) \;=\; -p\; e^{\mathrm{i}\Omega(p)}\sqrt{\frac{2\pi}{p\,(e^{2\pi p}-1)}}, \tag{A.47}$$

where we have defined $\Omega(p) = \mathrm{Arg}\,[\Gamma(\mathrm{i}p)]$. Equations (A.43) and (A.47) show that

$$V_o(k,p) \;=\; \frac{p}{\sqrt{2\pi}}\,\frac{e^{-\mathrm{i}\Omega(p)}\, e^{\mathrm{i}p\log|k|}}{|\,k\,|}\left[\frac{\theta(k)}{\sqrt{p\,(1-e^{-2\pi p})}} - \frac{\theta(-k)}{\sqrt{p\,(e^{2\pi p}-1)}}\right], \tag{A.48}$$

from which eq. (8.2) is deduced using eqs (5.37) and (A.41).

A.14 Proof of eq. (8.14)

Equation (6.21) is rewritten in the form

$$\bar{N}_o[f] \;=\; \int_0^\infty dp\, \tilde{f}\,(p) \int_0^\infty dp'\, \tilde{f}\,(p')^* \int_0^\infty \frac{dk}{2k}\, U(-k,p)\; U(-k,p')^*. \tag{A.49}$$

Using the expression (8.2) for the kernel of U and the formula

$$\int_0^\infty \frac{dk}{k}\; e^{\mathrm{i}\frac{(p-p')}{M}\log k} \;=\; 2\pi M\,\delta(p-p'), \tag{A.50}$$

eq. (8.14) is easily obtained from eq. (A.49).

A.15 Proof of eq. (8.18)

Using theorem 1 and eq. (8.2) we obtain

$$G(f_i \times f_j) = \langle A^* \tilde{f}_j^{\,*}, B\tilde{f}_i \rangle = 0, \tag{A.51}$$

$$G(f_i \times f_j^*) = \langle B\tilde{f}_j, B\tilde{f}_i \rangle = \int_0^\infty \frac{dp}{2p} \frac{\tilde{f}_j(p)^* \tilde{f}_i(p)}{e^{\frac{2\pi}{M}p} - 1}, \tag{A.52}$$

where $i, j = 1, 2, ..., n$. Expression (A.51) vanishes because of the presence in its kernel of the term $\delta(p + p')$. Theorem 2 then implies that

$$\bar{N}_o[f^{(n)}] = C^2 \sum_{\sigma \in \mathcal{P}_n} G(f_1 \times f^*_{\sigma(1)})\, G(f_2 \times f^*_{\sigma(2)}) \dots G(f_n \times f^*_{\sigma(n)}). \tag{A.53}$$

Noting that $G(f_i \times f_j^*) = G^{Th}_{\frac{2\pi}{M},out}(f_i \times f_j^*)$ (see eq. (7.28)), eq. (8.18) is deduced from theorem 4.

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