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Exponential Bounds for Continuum Eigenfunctions of N-Body Schrödinger Operators

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Abstract. For any non-threshold bound state of an N-body quantum system, we give a non-isotropic exponential bound in the form of a geodesic distance associated with a suitably modified Agmon metric.

1 Introduction

Eigenfunctions of typical N-body Schrödinger operators decay exponentially in all directions of the configuration space, provided the energy is not a threshold [4]. The rate of decay depends on the direction and is not known in general. – Using the isotropic upper bound due to Froese and Herbst in Agmon's approach, we obtain an improved non-isotropic bound in the form of a geodesic distance. Our result provides a generalization of Agmon's well-known result to continuum eigenfunctions with non-threshold energy.

Consider a system of N quantum particles in $\mathbf{R}^3$ interacting by two-body potentials which decay pointwise to zero as the interparticle distance increases. Let H denote the Schrödinger operator of the system with center-of-mass motion removed, and suppose ψ is an eigenfunction of H with energy E. If E is discrete then a well-known theorem of Agmon tells us that

$$|\psi(x)| \leq C_\varepsilon e^{-(1-\varepsilon)\rho_E(x)} \qquad \forall \varepsilon > 0 \ , \tag{1.1}$$

where $\rho_E(x)$ denotes the geodesic distance from x to the origin w.r.t. the metric $ds^2 = 2(\Sigma_x - E)dx^2$ [1]. Here $\Sigma_x \in [\inf \sigma_{ess}(H), 0]$ is a threshold and dx^2 depends on the masses.

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For E in the continuum $\rho_E(x)$ is not defined anymore because $\Sigma_x - E$ is then negative in some directions $x/|x|$.

In the present work we derive a bound similar to (1.1) for arbitrary eigenvalues in the case where an isotropic exponential bound is *a priory* given. The precise assumption is that

$$e^{(1-\varepsilon)\alpha|x|}\psi \in L^2 \qquad \forall \varepsilon > 0 \tag{1.2}$$

for some $\alpha > 0$. Using this and the method of proof for (1.1) we arrive at a non-isotropic bound $\rho_{E,\alpha}$, which, after the substitution

$$\Sigma_x \to \tilde{\Sigma}_x = \max(\Sigma_x, E + \alpha^2/2) \tag{1.3}$$

in Agmon's metric, is defined in the same way as ρ_E. Our bound $\rho_{E,\alpha}$ thus improves on $\alpha|x|$ in directions where $\Sigma_x - E > \frac{1}{2}\alpha^2$ and coincides with it elsewhere. If E is a discrete eigenvalue then $\alpha = 0$ in (1.2) is admissible as well and $\rho_{E,\alpha=0} \equiv \rho_E$. To justify our assumption we recall that (1.2) for non-threshold eigenvalues follows from a well-known theorem due to Froese and Herbst, obtained under a further decay assumption on the potentials [4] (see also [7, 5, 6]). This theorem says that $E + \frac{1}{2}\alpha^2$ is a threshold (or infinite), like Σ_x by the way, if α is the largest constant for which ψ obeys (1.2).

Similar results were previously obtained by Perry and Derezinsky [8, 3]. Perry studied polynomially bounded solutions of the Schrödinger equation $H\psi = E\psi$, i.e. $\psi \in L^2_{-s}(\mathbf{R}^n)$ for some $s > 0$ rather than $\psi \in L^2(\mathbf{R}^n)$, and he obtained that $e^{(1-\varepsilon)\rho}\psi \in L^2_{-s}$ where $\rho = \rho_{E,\alpha=0}$ in our notation. Derezinsky starts from an eigenstate which has an exponential bound g in a region bounding a cone in the configuration space. He then obtains an exponential bound for the eigenfunction in the cone which involves a geodesic distance as well as the function g.

2 Notations and Result

We work in the frame of generalized N-body quantum theory as presented for instance in [7, 5, 6].

An N-body quantum system is characterized by a triple (X, L, V), where X is a finite dimensional Euclidean space, L a finite family of subspaces of X, and V a potential in X. The family L contains $\{0\}$ and X, is closed under intersection, and the potential V has for each $a \in L$ a decomposition

$$V(x) = V^a(\pi^a x) + I_a(x) \tag{2.4}$$

into a potential V^a, depending only on the orthogonal projection $\pi^a x$ of x onto $a^\perp$, and an intercluster potential I_a which is subject to decay assumptions. For our purpose the following properties are convenient and sufficient:

(1) $V \in L^1_{\mathrm{loc}}(X)$ and V_- is $-\Delta/2$ form-bounded with bound smaller than 1.
(2) $I_a(x) \to 0 \qquad |x|_a \to \infty$

Here Δ denotes the Laplace-Beltrami operator with respect to the metric $g(x,y) = xy$ (inner product) in X, $V_-(x) := \max(-V(x), 0)$, and $|x|_a := \min_{b \not\supset a} |x^b|$. (1) and (2) ensure that the decomposition (2.4) is unique, and that $V^a \circ \pi^a$ has again property (1) in X.

The Hamiltonian of the system is formally given by

$$H = -\frac{1}{2}\Delta + V \qquad \text{in } L^2(X) ,$$

and in this paper defined as the unique self-adjoint operator associated with the closure of the form $\int dx \left(\frac{1}{2}|\nabla\varphi(x)|^2 + V(x)|\varphi(x)|^2\right)$ on $C_0^\infty(X)$. The cluster decomposition Hamiltonians $H_a = -\Delta/2 + V^a \circ \pi^a$ are defined analogously. We set $\Sigma := \inf \sigma_{ess}(H)$ and $\Sigma_a := \inf \sigma(H_a)$. The function Σ_x introduced above then equals $\Sigma_{m(x)}$ where $m(x) := \bigcap_{b \in L: x \in b} b$.

Theorem 2.1 *Suppose $H\psi = E\psi$ and $e^{(1-\varepsilon)\alpha|x|}\psi \in L^2(X)$ for all $\varepsilon > 0$, where $E < 0$ and $\alpha > 0$, or $E < \Sigma$ and $\alpha \geq 0$. Then*

$$e^{(1-\varepsilon)\rho_{E,\alpha}}\psi \in L^2(X) \qquad \forall \varepsilon > 0 ,$$

where $\rho_{E,\alpha}$, after the substitution $\Sigma_a \to \tilde{\Sigma}_a := \max(\Sigma_a, E + \frac{1}{2}\alpha^2)$ in the metric, is defined in the same way as Agmon's bound ρ_E.

Remarks. (1) Our proof employs an approximation argument which requires a non-trivial isotropic exponential bound. This is the reason for the condition $\alpha > 0$ in the case $E \geq \Sigma$. If $E < \Sigma$ one has the bound originally due to O'Connor, which, incidentally, is also needed in proofs of Agmon's result [1, 7].

(2) A pointwise bound like the one in (1.1) immediately follows from the theorem if one has a subsolution estimate [2, 1]. To prove such an estimate slightly stronger assumptions on V_- are sufficient (see [1, Theorem 5.1]).

Here we only sketch the idea of the proof. The details may be found in [5]. We shall call f an *exponential bound (of ψ)* if $e^{(1-\varepsilon)f}\psi \in L^2(X)$ for all $\varepsilon > 0$. Our main tool to obtain exponential bounds is the following lemma.

Lemma 2.2 *Suppose $H\psi = E\psi$, $f, J \in C^\infty(X)$, $J, \nabla J$ and ∇f are bounded, and $f \geq 0$. Then*

$$J\left(H - \frac{1}{2}|\nabla f|^2 - E\right) J \geq \delta J^2$$

for some $\delta > 0$ implies

$$\|Je^f\psi\| \leq const\ \|\chi(x \in supp(\nabla J))e^f\psi\| .$$

The constant depends on $\delta, J, \nabla J$ and ∇f.

Using this lemma with J being a smoothed characteristic function of the complement of cones containing the subspaces $a \in L$ for which $\Sigma_a \leq E + \frac{1}{2}\alpha^2$, we show that $f \geq 0$ is an exponential bound if

$$|\nabla f(x)|^2 \leq 2(\tilde{\Sigma}_a - E) \qquad |x^a| \leq \eta|x|,\ |x| \geq 1 \tag{2.5}$$

for all $a \in L$ and some $\eta > 0$. The condition (2.5) allows us to establish the assumption of the lemma for $(1-\varepsilon)f$, and furthermore it ensures that $f(x) \leq \alpha|x| + \text{const}$ in $\{x | J(x) \neq 1\}$ by choice of J. Therefore $(1-J)e^{(1-\varepsilon)f}\psi \in L^2$ by assumption (1.2) and hence $Je^{(1-\varepsilon)f}\psi \in L^2$ by the lemma. The theorem now follows by an approximation argument given in [7].

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