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Iggeret ha-Mispar by Isaac ben Solomon Ibn al-Aḥdab (Sicily, 14th century) (Part II)

by *Ilana Wartenberg**

Abstract

The first article presented the life and *ensemble d'oeuvres* of Isaac ben Solomon Ibn al-Aḥdab. This second article focuses on Isaac's mathematical text *The Epistle of the Number* (*Iggeret ha-Mispar*), and its main Arabic source from the 13th century *Talḫīṣ aʿmāl al-ḥisāb (A Compendium on Operations of Calculation)* by Ibn al-Bannāʾ. First, I will discuss the denomination of the Hebrew text. Then, I will describe the main Arabic source of the Epistle and the general mathematical cadre directly connected with it. This will be followed by an elaboration of the arithmetical and algebraic contents of the Epistle. I shall concentrate on its algebraic part.

The Hebrew Text and its Arabic Source *Talḫīs aʿmāl al-ḥisāb*

The Epistle of the Number did not originally bear any title or hint at a possible denomination. It is Tony Lévy who gave the text the title *The Epistle of the Number* (אגרת המספר), since Isaac tells the readers about his decision to translate the Arabic Epistle on the Science of the Number into Hebrew and elaborate on it. This resulted in the Hebrew manuscript at hand.[1]

Identifying the Arabic source of *The Epistle of the Number* was a difficult task since neither Ibn al-Bannāʾ's name nor the title *Talḫīṣ aʿmāl al-ḥisāb* were explicitly evoked in *The Epistle of the Number*.[2] In the exordium to the Hebrew Epistle, cited below, Isaac tells us the story of a certain epistle in Arabic. The only clue in our Hebrew text is a reference to the later Arabic commentary to this Epistle by the same

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1 TONY LÉVY, L'algèbre arabe dans les textes hébraïques (I), Un ouvrage inédit d'Isaac ben Salomon al-Aḥdab (XIV siècle), in: *Arabic Sciences and Philosophy* 1 (2003), p. 286.

2 Ibid.

Arab author, entitled *The Unveiler of the Veil,* (מסיר המסוה). This title translates into *Rafʿ al-ḥiǧāb* in Arabic, and it is the title of a known commentary by Ibn al-Bannā' on *Talḫīṣ aʿmāl al-ḥisāb*.[3] This detail enabled Tony Lévy to trace the main Arabic source of *The Epistle of the Number*, i.e., *Talḫīṣ aʿmāl al-ḥisāb*.[4]

Isaac ben Solomon ben Ẓaddik ben al-Aḥdab the Spaniard said: there was an Arab scholar, whose admirers asked to compose for them a short treatise containing all matters of the Science of the Number in a concise way. He responded to their request and composed a very short epistle. He was doing marvels in [presenting] its methods and in abbreviating it, and he sent it to them. When the Epistle reached them, it seemed to have transcended their comprehension. They asked him to write a commentary. Sensing their lack of knowledge, he reacted slyly and composed for them the commentary of the Epistle, so unusual and profound, only to be understood by those gifted with logic and understanding for the reason of matters. He named it *The Unveiler of the Veil*. The readers reacted by naming it *The Bringer Back of the Veil.* In return, he wrote to them the following: "I am committed to make the best effort to expose the subject from its basic sources, but I am not obliged to being understood by the wild beasts." Later on, the Epistle spread among the wise, for whom it was of the utmost beauty. Its nature spread amongst them and they dedicated to it many different commentaries, all very extensive. When I reached their countries, I was dwelling in the *Tents of Qedar*, that Epistle reached me. I studied it with of one of their wise men. I have also seen the commentary of its author and of others until I could reach its deepest layers and reveal all its secrets. [Fol. 1v: 1-8]

Talḫīṣ aʿmāl al-ḥisāb is a succinct book on rules of arithmetic with a chapter dedicated to algebra, and its length is less than twentieth of the *Epistle of the Number*. *Talḫīṣ aʿmāl al-ḥisāb* contains no commentaries or examples. Isaac, as he promised in the beginning to *The Epistle of the Number*, presented a lengthy book in Hebrew, but he cautioned his readers that he would omit a few matters present in the Arabic source, as one can read in the following passage.

I saw fit to omit a few parts from the Epistle, since, in my view, they are of no use. I shall point out to those places as I reach them so no one could catch me *in flagrante delicto*, [i.e.,] someone who will read the Epistle in the Arabic language. In the beginning, he[5] elaborated on how many parts his book is divided into, and for each part, how many chapters it contains, and for each

[3] The complete title of IBN AL-BANNĀ'S commentary is *Rafʿ al-ḥiǧāb ʿan wuǧūh aʿmāl al-ḥisāb*, i.e., *The Unveiling of the Veil on the Methods of the Operations of Calculation.*

[4] LÉVY, L'algèbre Arabe, p. 288.

[5] I.e., Ibn al-Bannā'.

chapter, how many categories. I have omitted all that, because this issue in a big treatise is of little use and in a short treatise it does not make any sense if what is meant to be short turns out lengthy. [Fol. 1v: 15-18]

However, my research shows that he barely did so in reality. A meticulous comparison between *The Epistle of the Number* and *Talḫīṣ aʿmāl al-ḥisāb* does not reveal any material lacunae in the Hebrew text. To be sure, the *Epistle of the Number* includes more than its Arabic source: numerical examples in abundance, verbal problems and commentaries, as well as a presentation of the place-value decimal system and wide range of algorithms.

The Mathematical Background

The Arabic treatise *Talḫīṣ. aʿmāl al-ḥisāb* belongs to a mathematical genre in the Arabic mathematical tradition which is called *ʿilm al-ḥisāb*, i.e., *the science of calculation.* The tradition of *ʿilm al-ḥisāb* (or *ḥisāb* in short) spans approximately from the 8th through to the 15th century.[6] *Ḥisāb* books deal mainly with arithmetic, i.e., the branch which concerns numeration systems and operations upon known quantities: addition, subtraction, multiplication, division, extraction of a square or even higher roots, sexagesimal calculation, as well as theoretical and practical problems. Some *ḥisāb* books show how to sum a finite series of the natural numbers, even and odd numbers, and the squares and cubes of natural numbers. However, the contents of *ḥisāb* books extend beyond mere arithmetic, and they often include a chapter on algebra,[7] which

[6] PASCAL CROZET, Aritmetica, in *Storia della scienza* (Instituto della enciclopedia), vol. III; ROSHDI RASHED (ed.), *La civiltà islamica*, Rome 2002, pp. 498-506.

[7] Algebra is first presented systematically in Al-Khwārizmī's book *Kitāb al-muḫtasar fī ḥisāb al-ğabr wal-muqābala*, which was written between 813 and 833 in Baghdad. This book marks the official birth of a new discipline. It presents the algebraic operations of *al-ğabr* (*restoration*) and *al-muqābala* (*opposition*), and the algebraic objects (*dirham*, *thing*, *estate*). For the first time in history, one finds a presentation of the six canonical equations. Al-Khwārizmī presented algorithms to solve linear and quadratic equations and provides geometric proof of these procedures. He then uses algebra to solve practical problems of heritage distribution, land measurement and commercial transactions. The algebra within the *ḥisāb* tradition is characterised by its practical nature: mainly, the solution of equations of the first and second degree, the formation of algebraic expressions by the algebraic "atoms" (numbers, things, estates). In *ḥisāb* books, there is no concern for proof, theoretical or philosophical discussion, but rather an emphasis on how to perform the calculation. See MOHAMED BEN MŪSA, *The*

provides tools to find the unknown by the solution of equations, alongside arithmetical methods such as *the double false position* (the method of scales) or *the rule of three.*

Around the middle of the 14th century, the famous Moroccan mathematician, Abu l-ʿAbbās Aḥmad ibn Muḥammad ibn Uṯmān al-ʿAzdī ibn al-Bannāʾ (1256-1321), composed a concise treatise on *ʿIlm al-Ḥisāb* named *Talḫīṣ aʿmāl al-ḥisāb*. This book presents various calculation procedures in a precise and condensed manner and is divided in two parts. The first part deals with arithmetical operations on numbers: *integers*, *fractions* and *square roots*.[8] The second part presents rules and procedures which "allow us to find the value of the sought unknown from the given known": the rule of three, double false position and algebraic procedures such as the solution of equations of the first and second degree. There are no demonstrations for the proposed mathematical procedures in *Talḫīṣ aʿmāl al-ḥisāb.*

Talḫīṣ aʿmāl al-ḥisāb was a well-known book in the Arabic mathematical realm, and many commentaries were written on it. Among these commentaries, one finds *Rafʿ al-ḥiǧāb*, which was composed by Ibn al-Bannā' himself, as mentioned above. However, *Rafʿ al-ḥiǧāb* does not include all subjects treated in *Talḫīṣ aʿmāl al-ḥisāb* and it treats many other subjects which are not found in the *Talḫīṣ*.[9]

The Epistle of the Number: The Highlights of its Mathematical Contents (Books I and II)

Book I deals with the basic arithmetical operations, i.e., addition, subtraction, multiplication and division, applied on *known quantities*, i.e., numbers. The types of numbers encountered are integers, fractions and roots. Book II focuses on *unknown quantities* and the ways to find their values either by arithmetical or by algebraic methods. The mathematical part of *The Epistle of the Number* starts with the definition of the number.

The number is a multitude composed of units. [Fol. 1v: 18-19]

Algebra of Moḥamed ben Musa, ed. & transl. by ROSEN FREDERIC, London 1831, pp. v-xii, ROSHDI RASHED, L'algèbre, pp. 31-34, by the same author, *Entre Arithmétique et Algèbre*, pp. 17-20, and AHMAD DJEBBAR, *L'algèbre arabe, genèse d'un art*, Paris 2005, p. 23-32.

[8] Including irrational, i.e., *inexpressible*, quantities.

[9] MOHAMED ABALLAGH, Les fondements des mathématiques à travers le *Rafʿ al-Ḥiǧāb* d'Ibn al-Bannāʾ (1256-1321), in: *Histoire des mathématiques arabes*, Alger 1988, p. 138.

Isaac emphasizes the fact that 1 is not a number, telling us that this statement or a similar one can be found at Ibn Rushd's *Physics*.

[…] out of all the comments made here, there is not a better one, in his view, than the one which he mentioned, i.e., that the one is not a number. You will find this comment or a similar one in Ibn Rushd's *Physics*. [Fol. 1v: 22-23]

This fact is later corroborated in the presentation of the multiplication table, since there one sees that 1 does not change the value of the number it multiplies.

When multiplying a number by 1 or 1 by it, the number remains as is and is not multiplied. Commentary: his saying concerns the quick multiplication and he began by the multiplication of the one. He said that when one is multiplied by whichever number, nothing is added or subtracted. Also, when a number is multiplied by it, i.e., by one, when multiplying 1 by 2, it is 2, by 3, it is 3, by 4, it is 4 and so with the rest and this is evidence that the one is not a number! [Fol. 12v: 30-33]

Nevertheless, shortly afterwards, one sees that 1 is treated as the first of all numbers:[10]

There are odd [numbers] which cannot be divided at all but by one, such as 5, 7, 11 and 13. These numbers are numerous and they are called *prime numbers* because they are not divisible but by the one, which is primary of all numbers. [Fol. 2r: 1-2]

Isaac then discusses the nature of the mathematics he is dealing with in *The Epistle of the Number*. He accentuates its practical, non-philosophical, nature. In the context of the *ʿilm al-ḥisāb*, Isaac's remark comes as no surprise, since the core of the *ḥisāb* tradition involves practical tools for arithmetical calculation. Isaac emphasizes the similarity between the roles of the arithmetician and the doctor, neither of whom searches the nature of the subjects they treat, i.e., numbers and people, respectively. Both are concerned with practical matters only.

I find this correct, since the master of this science need not verify and announce the nature of the number, just as the doctor does not have to verify the nature of Man; and in this science, one does not enquire the quiddity of the number. However, we scrutinize the number in respect to how it adds, divides, multiplies, increases, decreases, and so on […] [Fol. 1v: 24-25]

Then, while comparing arithmetic to geometry, Isaac tells us more about the objects of arithmetic:

The Science of the Number does not consider the attributed objects, but the attributes themselves; in the same way that geometry has no concern as to whether

[10] It is well known that even in the Greek tradition, 1 was treated ambivalently, i.e., both as a *number* and a *non-number*.

the triangle is made of wood or brass, since it considers a simple triangle as having no material, and so does the Science of the Number. [Fol. 13v: 10-12]

Next, Isaac presents the categorisation of numbers into the following subsets: *integers* and *fractions*. Integers are further subdivided into *even* and *odd* numbers. Even numbers are further subdivided into *even-times even*, *even-times odd* and *even-times even-times odd*. Odd numbers are subdivided into *prime* and *odd-times odd* numbers.

Then, the three basic ranks are presented: *units*, *tens* and *hundreds*. These ranks are the building stones for all numbers. In his commentary, Isaac first explains the place-value decimal system graphically and then he describes how to write numbers with their corresponding ranks.

Isaac said: in any case, one should elaborate here to make known the [written] figures and its ranks… here is their figure, which I draw to you in writing 1 2 3 4 5 6 7 8 9. If you wish to have different figures, use each one of these, which alone stands for units, as 1 indicates one… whenever seeing one of these figures in the second rank, it indicates tens, such as when the rank has a number, for example, 21… if the first [rank] does not have a number, they used to draw there a circle, which they call *sifr*, which indicates the rank, after which a number is set, such as 20. Here the two represents twenty because it is in the second rank. If we had two zeros, the two would be two hundred… 2000, the two are units of one thousand, and from there one revolves to units […] [Fol. 2r: 19-27]

Three systems of numeration are simultaneously used throughout the *Epistle of the Number*. The first one is the place-value decimal system with Arabic numerals, which was presented above. This system is never used in the *Epistle of the Number* with Arabic numerals, though only with the first nine Hebrew letters א ב ג ד ה ו ז ח ט, and 0. The second system is a Hebrew *ABJAD*, literally ABCD. This is an alpha-numerical system, whereby each letter of the alphabet represents a numerical value. In an equivalent English system, the letter A has the value 1, J-10, S-100, etc.[11] The third system used in *The Epistle of the Number* is the rhetoric usage of numbers, i.e., numbers are expressed in words. Often, more than one system is used in the same context, in particular in the examples.

In the next section, the reader is taught how to read long written numbers and how to write numbers given rhetorically. A one-to-one relation is established between a number, its name and its rank.

The next chapter is dedicated to the addition of integers.

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[11] AHMAD S. SAIDAN, Numération et arithmétique, in: *Histoire des Sciences Arabes*, Paris 1997, vol. II, p. 12. The Hebrew alphabet ends with the letter ת, which represents 400.

The addition is the collection of numbers to each other until reaching a unique expression. There are five types [of addition]. [Fol. 2v 36-37]

The five different types of addition elaborated in our text are: the addition of numbers which bear no ratio, addition of the geometric sequence 1, 2, 4…, the addition of the sequence of the natural numbers, odd and even numbers, the squares of the natural numbers, even and cubes as well as the cubes of natural, odd and even numbers. The reader learns that the result of adding numbers with no ratio can be verified by subtraction.

The test of the addition by the subtraction of [the number in] one of the lines from the result, and the [number in the] other line [should] remain. [Fol. 3r: 9]

However, Isaac states the pedagogical problem of this statement at this point, where no subtraction has been taught yet. The solution to this difficulty lies in the claim that a student is supposed to learn the entire arithmetical theory, which includes both addition and subtraction, and only then, will he know how to use the test of subtraction.

[…] one should know this science in its whole, in as much as possible […] [Fol. 3v: 9-10][12]

For the other types of addition the appropriate rules are given. The next chapter concerns subtraction.

Subtraction is the problem of seeking the remainder after casting one of the numbers from the other. [Fol. 6v: 1]

There are two types of subtraction here: the first one is a one-time subtraction, carried out in a similar way to the addition of numbers which bear no ratio. Just like before, verification of the solution is done by adding the result to the subtractive and obtaining the number subtracted from. The second type of subtraction is the casting by 7, 8, and 9.[13] This is mainly used as a verification test for addition, subtraction and multiplication.

A chapter on multiplication follows.

The multiplication is an allusion for the repetition of one of the numbers by the rate of the units in the second one […] [Fol. 10r: 20-21]

There are three types of multiplication. The first one is by *translation* (either *lying* or *standing*), and it is used for any pair of numbers. The second type of multiplication is by *semi-translation*, which can only be used for two identical numbers. The third multiplication is *without translation*, and it is carried out with tables with diagonal lines. Then, the multiplication of all numbers between 1 and 9 (the multiplication

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[12] האדם צריך שידע החכמה בשלמותה כפי האפשר

[13] i.e., modulo 7, 8, 9.

table) is presented, accompanied by a graphical presentation in form of a table. Division is the subject of the next chapter.

Division is the decomposition of the divided [number] into equal parts [with the number of parts the same as] in the number of units within the divisor. [Fol. 13r: 22-23]

There are two types of division: division of a small number by a larger one, which is called *denomination*, and the division of a larger number by a smaller number, which is referred to as *simple division*. The text first gives the algorithm for the second type of division. As for the first type, the reader is told that denomination results in a fraction; this subject is treated in a later chapter in the Epistle. Factorization of the dividend and the divided term is used for finding the denomination and for resolving problems of proportional divisions. The latter presents a class of problems, common in the Arabic mathematical tradition, which concern the uneven distribution of heritage. Isaac mentions the Arabic context while emphasizing that it is the mathematical method used in these problems that is relevant for the Hebrew text, and not the Muslim inheritance laws per se.

This is often used by the Muslims in their heritage… all relatives are heirs, but [they inherit] in unequal [parts]. The father, for instance, inherits one half, the son one third, the brother one fourth, the sister one fifth… and this is just an example, they do not necessarily inherit according to this rule. This [i.e., heritage per se] is not our concern, this is only an example […] [Fol. 15r: 11-12]

Then, he explains the distribution of sums, which is followed by various divisibility rules. Isaac also discusses the division of continuous quantities by continuous ones, discrete ones by discrete ones and continuous ones by discrete ones. This shows Isaac's familiarity with Aristotelian categories. The last part concerns Eratosthenes' sieve. The following chapter deals with *restoration* and *reduction* of numbers, which is carried out by multiplication and division, respectively.

The restoration is the reparation and the reduction is its opposite. The intention in restoration and reduction is the knowledge of what number is to be multiplied by a certain number such that the required number results. Restoration can only be done from the smaller to the larger, and reduction is its opposite. [Fol. 17r: 34-37]

So far, the text has dealt with arithmetical operations upon integers. The second part of Book I discusses arithmetical operations on fractions. First, Isaac defines fractions:

Fractioning is the ratio between two numbers, whether in part or in parts. The ratio between the part and its denomination is called *a fraction*. [Fol. 17v: 12-13]

In this part, there are six chapters which elaborate on various aspects of fractions and possible operations upon them. Also, the fraction line is introduced. The six chapters are: the naming of fractions and the determination of their numerator, the addition and subtraction of fractions, the multiplication of fractions, the division of one fraction by another, the restoration and reduction of fractions and finally, the conversion of fractions.

The third part of Book I concerns the approximation of square roots. It is followed by methods to find the root of *binomials* and *apotomes*, i.e., sums and differences of numbers, roots or roots of roots. Next, multiplication of roots is explained, followed by their division. [14] This ends Book I.

Book II opens with the presentation of two arithmetical methods to find the value of the unknowns: the *rule of three* and the *method of scales*, i.e., the *double false position*. The *rule of three* is applied to a sequence of *four proportional numbers*, in which one of the numbers is unknown. The result is the value of the unknown number. *The rule of three* is not presented here per se, but as a concrete means to solving linear equations. The *method of scales* involves guessing two possible solutions or only a single guess, in case one happened to guess the right solution at the very start. The correct solution is reached by an arithmetical manipulation on the errors. All arithmetical operations required in the application of these two methods have already been elaborated on in Book I.

The second part of Book II elaborates algebraic notions and algorithms as well as with arithmetical operations upon algebraic expressions. The *Epistle of the Number* presents the algebraic *species*: the *number* (already encountered in Book I), the *root* (or the *thing*), which is the unknown, the *square* (or the *estate*), the *cube*, the *square square*, etc., up to the *cube cube cube*. The basic algebraic operations are *restoration, opposition* and *equalisation*. These three operations help revert any given equation to one of the six canonical equations.[15] Our text presents the six

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[14] The appearance of *binomials* and *apotomes* here is very important, because apart from the translation of the Book X of *The Elements* and Ibn Ḥaytham's commentary, this is the only Hebrew text known to us which discusses arithmetical operations upon irrational quantities. See TONY LÉVY, The Establishment of the Mathematical Bookshelf of the Medieval Hebrew Scholar: Translations and Translators, in: *Science in Context* 10 (1997), pp. 431-451.

[15] MOHAMED BEN MŪSA, *The Algebra of Moḥamed ben Musa*, ed. & transl. ROSEN FREDERIC, London 1831, pp. v-xii.

canonical equations with their corresponding solution algorithms, while discussing the procedure of normalization of the equations.[16]

Furthermore, two forms of abbreviated notation for the algebraic species is given. It is also explained that equations of the third degree or a higher one cannot be solved unless no numbers are present and the equation is reducible to a quadratic equation. Algebraic expressions are formed by the addition and subtraction of algebraic species. The addition and subtraction of dissimilar species is carried out by the *particle of addition* and the *particle of subtraction*, respectively, since no real operation can take place.

Next, the multiplication of algebraic expressions is carried out with the help of tables, followed by the division of algebraic expressions. However, only the division by a monomial is permissible, and one must never divide a lower species by a higher one.

The last three folios of *The Epistle of the Number,* truncated at the end of folio 38v, deal with a series of problems of practical nature and problems which are reduced to the solution of linear or quadratic equations: six problems which reduce to the solution of the six equations, tricks to overcome problems which involve the division of a lower species by a higher one, one charity calculation, two distance problems and five problems of horse purchase by several associates, as well as a beginning of a sixth problem, at which point the *unicum* of the Epistle is truncated.

A Closer Look at Algebra

The three basic algebraic species presented first are *numbers* (מספרים), *things* (דברים; also called *roots* שרשים) and estates (ממונות; also called *squares* מרובעים). All three species have a numerical value, which is known a-priori for numbers and is unknown for roots and squares, and hence, needs to be found. Numbers, roots and squares constitute the three building stones of further species in the same way that units, tens and hundreds can form any number. Here, the aim is to reduce the species to numbers, roots and squares, and as is seen later, only numbers, roots and squares are permitted in an equation in order for it to be solvable.

Isaac subdivides the three species also into two categories. Numbers belong to a different category than roots and squares for the following

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16 Ilana Wartenberg, The Epistle of the Number *by Isaac ben Salomon Ibn al-Aḥdab (Sicily, 14th century) - an Episode in Algebra in Hebrew,* Ph. Diss. Paris / Tel-Aviv 2007, pp. 45-53.

reasons. First, numbers possess only one denomination, whereas things and estates have a second name (roots and squares, respectively). The second denomination emphasizes the dependent, i.e., interrelated nature of roots and squares, i.e., roots are always of squares and squares are always of roots.

Numbers, on the other hand, bear a unique, independent name, which is unrelated to other species, and hence, derives its identity intrinsically. The number always has a known value, unlike roots and squares, which can be set simply. A second layer of context separates numbers from roots and squares in the following sense: every number can be a root (to its square) and every square has a root, either rational מדובר or not, לא מדובר, but a number is such in its own value.

Next, further species are given in order, up to a cube cube cube: a square is the multiplication of the root by itself, in modern notation we write: $x^2 = x \cdot x$. The multiplication of the root by a square yields a cube, $x^3 = x \cdot x^2$. When multiplying a root by a cube or a square by a square, a *square square* is obtained $x^4 = x \cdot x^3 = x^2 \cdot x^2$, up to a *cube cube cube* $x^9 = x^3 \cdot x^3 \cdot x^3$. *The Epistle of the Number* goes as far as a cube cube cube in its definitions without any explicit generalization for higher powers.

Upon these species algebraic operations are performed: *restoration*, *opposition* and *equalization* constitute the basic operations of algebra, and the ensemble of the three operations is referred to as *restoration*, i.e., *al-ğabr.*[17] Restoration was already mentioned in Book I in an arithmetical context. There, deficient numbers were restored by addition or multiplication. Here, the very same operation can be used for all algebraic species. A square less a root is *restored*, i. e., completed, to a whole square by the addition of the root. One can write it in the following notation: *restoration* $(x^2 - x) \rightarrow x^2$.

In *The Epistle of the Number*, the three algebraic operations are divided into two categories. The first category includes *restoration* and *opposition*, and the second one *equalization*. These operations are distinguished in terms of dependence or independence in the following sense: restoration and opposition belong in the same category because they are both independent procedures, i.e., restoration can be carried

[17] See MOHAMED BEN MUSA, *The Algebra*, pp. 5-21 and ROSHDI RASHED, L'algèbre, pp. 31-34. One notes, however, that equalization is not mentioned by al-Khwārizmī, only restoration and opposition. For a detailed discussion of the algebra within *The Epistle of the Number*, see WARTENBERG, *The Epistle*, pp. 54-100.

out by itself without opposition, equalization or any other operations. Similarly, opposition does not necessitate either restoration or equalization, as one can easily infer from the examples given in the definition. On the other hand, equalization cannot be done without either restoration or opposition *par definition*.

The Core of Algebra: The Six Classical Equations

Isaac presents the six canonical equations, in which combinations of numbers, roots and squares are set equal. There are three *simple* equations, which are equations which include only two species and there are three *composite* equations, which involve all three species.

I. He said: the first of the simple form, according to the general convention, is estates equals things. [Fol. 28v: 37]

וראשון הנפרדים לפי מה שתרוץ עליו ההסכמה: ממונות ישוו דברים.

II. The second part of the simple type: estates equal numbers. [Fol. 29r: 10]

אמר: והחלק השני מן הנפרדים: הממונות ישוו המספרים.

III. The third [type] is roots equal numbers. [Fol. 29r:14-15]

והשלישי, שרשים ישוו מספרים.

IV. [As for] the three composite [equations], in the first [composite case], [i.e.,] in the fourth [equation], the number is isolated. [Fol. 29r:18]

והשלשה מורכבים- הראשון מהם- החלק הרביעי יתייחד בו המספר.

V. [In] the fifth [equation], the root is isolated. [Fol. 29r:23]

והחמשי יתייחד בו השרש.

VI. The sixth [type of equation is the one in which] the square is isolated. [Fol. 29r:29]

והששי יתייחד בו הממון.

In modern notation, one could carefully[18] write the six equations as follows:

I. (Simple) Squares equal Numbers $ax^2 = bx$

II. (Simple) Squares equal numbers $ax^2 = c$

III. (Simple) Things equal numbers $bx = c$

IV. (Composite) Squares and things equal numbers $ax^2 + bx = c$

V. (Composite) Estates/Squares and numbers equal things $ax^2 + c = bx$

VI. (Composite) Things and numbers equal an estate $bx + c = x^2$

[18] I say *carefully*, since one must bear in mind that in our Hebrew text there is no trace of the modern algebraic symbolism.

A Numerical Example

4 squares less 2 things equals 3 squares and 2 things. Add the 2 subtractive things to become 4 integer squares. Equalize again, by adding the two things to the two things in the other term. It becomes 4 squares equals 3 squares and 4 things. Then, one must equalize again and oppose, because there are squares on both terms. Therefore, subtract 3 squares from the 4 squares and a square equalling 4 things remains.[19] [Fol. 28v: 5-8]

$$4x^2 - 2x = 3x^2 + 2x$$

Restauration $(4x^2 - 2x) \rightarrow 4x^2 - 2x + 2x = 4x^2$

Equalization $(3x^2 + 2x) \rightarrow 3x^2 + 4x$

$\rightarrow 4x^2 = 3x^2 + 4x$

Opposition $(4x^2 = 3x^2 + 4x) \rightarrow 4x^2 - 3x^2 = 3x^2 - 3x^2 + 4x$

$\rightarrow x^2 = 4x$

Summary

This article gave an overview of the mathematical contents of *The Epistle of the Number*, and allowed a deeper glimpse into basics of algebra within. As emphasized in the first article, *The Epistle of the Number* is the first text known to us which includes elaborate algebraic materials within the medieval Hebrew mathematical corpus. This article also aimed to complete the contextual elements which were provided in the first article on Isaac's life and the historical circumstances in which he lived. Through the analysis of the mathematical contents of *The Epistle of the Number*, one discovers the richness of arithmetical and algebraic scope of this rare testimony of algebra in medieval Hebrew,[20] with its rich novel algebraic vocabulary.[21] At the same time, *The Epistle of the Number* is a precious copious treatise on arithmetic.

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[19] משל אחר: ד' ממונות פחות ב' דברים ישוו ג' ממונות וב' דברים. תוסיף ב' דברים החסרים יהיו ד' ממונות שלמים. ותשוב להשוות, ותוסיף הב' דברים עם הב' דברים שבצד השני, יהיו ד' ממונות ישוו ג' ממונות וד' דברים. ובזה אתה צריך לשוב ולהשוות ולהקביל, בעבור כי יש ממונות בשני הצדדין. ולכן, תגרע ג' ממונות מן הד' ממונות וישאר ממון ישוה ד' דברים.

[20] For the description of previous traces of algebra in Hebrew which were discovered and researched by TONY LÉVY, see WARTENBERG, *The Epistle*, pp. 7-12.

[21] The novel algebraic vocabulary is described in the lexicon of my dissertation. See ibid, pp. 101-147.