A theorem in operational calculus

Autor(en): **Dahiya, R.S.**

Objekttyp: **Article**

Zeitschrift: **Mitteilungen / Vereinigung Schweizerischer Versicherungsmathematiker = Bulletin / Association des Actuaires Suisses = Bulletin / Association of Swiss Actuaries**

Band (Jahr): **64 (1964)**

PDF erstellt am: **22.07.2024**

Persistenter Link: https://doi.org/10.5169/seals-966968

A Theorem in Operational Calculus

By R. S. Dahiya, Pilani (Rajasthan), India

Summary

The author proves a theorem in the field of Laplace transformations. Two examples show the possible applications.

1. Introduction. The integral equation

$$\Phi(p) = p \int_0^\infty e^{-pt} f(t)\, dt, \qquad R(p) > 0. \quad (1.1)$$

represents the classical Laplace transform and the functions $\Phi(p)$ and $f(t)$ related by (1.1), are said to be operationally related to each other. $\Phi(p)$ is called the image and $f(t)$ the original. Symbolically we can write

$$\Phi(p) \doteqdot f(t) \qquad \text{or} \qquad f(t) \doteqdot \Phi(p).$$

The transforms, we have dealt with, are either the Hankel transform or another transform in which the kernel is designated by $\tilde{\omega}_{\mu,\nu}(x)$, where

$$\tilde{\omega}_{\mu,\nu}(x) = \sqrt{x} \int_0^\infty J_\nu\left(\frac{x}{t}\right) J_\mu(t) \frac{dt}{t}. \quad (1.2)$$

The integral on the right was first evaluated by C. V. H. Rao [1] and that it plays the role of a transform was conjectured by Watson [2]. Later on Bhatnagar has proved in detail that it plays the role of a transform. In this paper, we make use of Goldstein's theorem [3] as follows:

Let (i) $\Phi_1(p) \doteqdot f_1(t)$,

(ii) $\Phi_2(p) \doteqdot f_2(t)$,

then after applying Goldstein's theorem, we get

$$\int_0^\infty f_1(t)\,\Phi_2(t)\,\frac{dt}{t} = \int_0^\infty \Phi_1(t)\,f_2(t)\,\frac{dt}{t},$$

where $\Phi(p)$ is the image and $f(t)$ is the original.

2. Theorem. Let

(i) $\psi(p) \doteqdot f(x)$,

(ii) $t^{\mu-\lambda-\frac{1}{2}} f(t)$ be $R_{\mu,\nu}$,

then

$$x^{2\mu-\lambda} f(x) \doteqdot \frac{2^{2\mu-\lambda+1}\,\Gamma\left(\mu+\frac{3}{2}\right)\Gamma\left(\frac{\mu+\nu+3}{2}\right)}{\Gamma\left(\frac{\lambda+3}{2}\right)\Gamma\left(\frac{\lambda}{2}+2\right)\Gamma\left(\frac{\nu-\mu-1}{2}\right)}$$

$$\cdot\, p^2 \int_0^\infty t^{\lambda+1}\psi(t)\; {}_3F_2\left(\begin{matrix}\mu+\frac{3}{2},\ \frac{\mu+\nu+3}{2},\ \frac{\nu-\mu+3}{2}; \\ \frac{\lambda+3}{2},\ \frac{\lambda}{2}+2\,; \end{matrix}\ t^2 p^2\right) dt, \quad (2.1)$$

provided $f(x)$ is absolutely integrable in $(0,\infty)$, $t^{\mu-\lambda-\frac{1}{2}} f(t)$ and $x^{2\mu-\lambda} f(x)$ are integrable in $(0,\infty)$, and $R(\mu) \geqq -\frac{1}{2}$, $R(\nu) \geqq -\frac{1}{2}$, $R(\lambda+2) \leqq 0$.

Proof: Let

$$p^m\,\tilde{\omega}_{\mu,\nu}(p) \doteqdot F(x)$$

then

$$x^\lambda F\left(\frac{1}{x}\right) \doteqdot p^{\frac{1-\lambda}{2}} \int_0^\infty J_{\lambda+1}(2\sqrt{xp})\, x^{\frac{\lambda-1}{2}+m}\,\tilde{\omega}_{\mu,\nu}(x)\,dx, \qquad (2.2)$$

$R(\lambda+2) > 0$, $R(\lambda+m+\mu) \geqq -\frac{3}{2}$, $R(\lambda+m+\nu) \geqq -\frac{3}{2}$, $R(\lambda+2m) < 0$

or,
$$x^\lambda F\left(\frac{1}{x}\right) \doteqdot p^{\frac{1-\lambda}{2}} \int_0^\infty J_{\lambda+1}(2\sqrt{pt})\, t^{\frac{\lambda-1}{2}+m}\,\tilde{\omega}_{\mu,\nu}(t)\,dt, \qquad (2.3)$$

or, $$x^{\lambda} F\left(\frac{1}{x}\right) \doteqdot \frac{p^{1-m-\lambda}}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma(\lambda+m+s)}{\Gamma(-m-s+2)} 2^{1-2s} \frac{\Gamma\left(\frac{\mu-s}{2}+\frac{3}{4}\right)\Gamma\left(\frac{\nu-s}{2}+\frac{3}{4}\right)}{\Gamma\left(\frac{\mu+s}{2}+\frac{1}{4}\right)\Gamma\left(\frac{\nu+s}{2}+\frac{1}{4}\right)} p^{-s}\, ds\,, \tag{2.4}$$

or, $$x^{\lambda} F\left(\frac{1}{x}\right) \doteqdot \frac{p^{1-\lambda-m}}{2\pi i} \int_{c-i\infty}^{c+i\infty} 2^{\lambda+2m-1} \frac{\Gamma\left(\frac{\lambda+m+s}{2}\right)\Gamma\left(\frac{\lambda+m+s+1}{2}\right)\Gamma\left(\frac{\mu-s}{2}+\frac{3}{4}\right)}{\Gamma\left(\frac{-m-s+2}{2}\right)\Gamma\left(\frac{-m-s+3}{2}\right)\Gamma\left(\frac{\mu+s}{2}+\frac{1}{4}\right)}$$

$$\cdot \frac{\Gamma\left(\frac{\nu-s}{2}+\frac{3}{4}\right)}{\Gamma\left(\frac{\nu+s}{2}+\frac{1}{4}\right)} p^{-s}\, ds\,. \tag{2.5}$$

By putting $m = \mu - \lambda + \frac{1}{2}$ and then evaluating the integral, we get

$$x^{\lambda} F\left(\frac{1}{x}\right) \doteqdot 2^{2\mu-\lambda+1} p^{2} \sum_{n=0}^{\infty} \frac{(-1)^n}{(n)!} \frac{\Gamma\left(\mu+n+\frac{3}{2}\right)\Gamma\left(\frac{\mu+\nu+3}{2}+n\right)}{\Gamma\left(\frac{\lambda}{2}+n+\frac{3}{2}\right)\Gamma\left(\frac{\lambda}{2}+n+2\right)\Gamma\left(\frac{\nu-\mu}{2}-n-\frac{1}{2}\right)} p^{2n} \tag{2.6}$$

or, $$x^{\lambda} F\left(\frac{1}{x}\right) \doteqdot 2^{2\mu-\lambda+1} p^{2} \frac{\Gamma\left(\mu+\frac{3}{2}\right)\Gamma\left(\frac{\mu+\nu+3}{2}\right)}{\Gamma\left(\frac{\lambda+3}{2}\right)\Gamma\left(\frac{\lambda}{2}+2\right)\Gamma\left(\frac{\nu-\mu-1}{2}\right)}$$

$$\cdot\, {}_3F_2\left(\begin{matrix}\mu+\frac{3}{2}, \frac{\mu+\nu+3}{2}, \frac{\mu-\nu+3}{2}; \\ \frac{\lambda+3}{2}, \frac{\lambda}{2}+2;\end{matrix}\; p^2\right). \tag{2.7}$$

On writing $\dfrac{x}{t}$ for x we have

$$\left(\frac{x}{t}\right)^{\lambda} F\left(\frac{t}{x}\right) \doteqdot 2^{2\mu-\lambda+1} \frac{\Gamma\left(\mu+\frac{3}{2}\right)\Gamma\left(\frac{\mu+\nu+3}{2}\right)p^2\,t^2}{\Gamma\left(\frac{\lambda+3}{2}\right)\Gamma\left(\frac{\lambda}{2}+2\right)\Gamma\left(\frac{\nu-\mu-1}{2}\right)}$$

$$\cdot\, {}_3F_2\left(\begin{matrix}\mu+\frac{3}{2},\ \frac{\mu+\nu+3}{2},\ \frac{\mu-\nu+3}{2};\\ \frac{\lambda+3}{2},\ \frac{\lambda}{2}+2;\end{matrix}\ t^2\,p^2\right). \qquad \text{(A)}$$

Let (i) $\quad p^{\mu-\lambda+\frac{1}{2}}\,\tilde{\omega}_{\mu,\nu}(p) \doteqdot F(x)\,, \qquad$ (B)

(ii) $\quad \psi(p) \doteqdot f(x)\,, \qquad$ (C)

Also from (B), $\quad (ap)^{\mu-\lambda+\frac{1}{2}}\,\tilde{\omega}_{\mu,\nu}(ap) \doteqdot F\left(\dfrac{x}{a}\right). \qquad$ (D)

Applying Goldstein's theorem to (C) and (D), we get

$$\int_0^\infty f(t)\,(at)^{\mu-\lambda+\frac{1}{2}}\,\tilde{\omega}_{\mu,\nu}(at)\,\frac{dt}{t} = \int_0^\infty \psi(t)\,F\left(\frac{t}{a}\right)\frac{dt}{t}. \qquad (2.8)$$

On writing x for a and multiplying by x^{λ} to both sides, it follows

$$x^{\lambda}\int_0^\infty f(t)\,(xt)^{\mu-\lambda+\frac{1}{2}}\,\tilde{\omega}_{\mu,\nu}(xt)\,\frac{dt}{t} = \int_0^\infty \psi(t)\,F\left(\frac{t}{x}\right)\left(\frac{x}{t}\right)^{\lambda} t^{\lambda-1}\,dt. \qquad (2.9)$$

Interpreting with the help of (A), we get

$$x^{\mu+\frac{1}{2}}\int_0^\infty f(t)\,t^{\mu-\lambda-\frac{1}{2}}\,\tilde{\omega}_{\mu,\nu}(xt)\,dt \doteqdot \frac{2^{2\mu-\lambda+1}\,\Gamma\left(\mu+\frac{3}{2}\right)\Gamma\left(\frac{\mu+\nu+3}{2}\right)}{\Gamma\left(\frac{\lambda+3}{2}\right)\Gamma\left(\frac{\lambda}{2}+2\right)\Gamma\left(\frac{\nu-\mu-1}{2}\right)}$$

$$\cdot\, p^2\int_0^\infty t^{\lambda+1}\,\psi(t)\,{}_3F_2\left(\begin{matrix}\mu+\frac{3}{2},\ \frac{\mu+\nu+3}{2},\ \frac{\mu-\nu+3}{2};\\ \frac{\lambda+3}{2},\ \frac{\lambda}{2}+2;\end{matrix}\ t^2\,p^2\right)dt. \qquad (2.10)$$

If $t^{\mu-\lambda-\frac{1}{2}}f(t)$ is $R_{\mu,\nu}$, we obtain the required result.

3. Example 1. Let $t^{\frac{\mu-\nu+1}{2}} J_{\frac{\mu+\nu}{2}}(t)$ be $R_{\mu,\nu}$,

then $f(t) = t^{\lambda - \frac{\nu+\mu}{2}+1} J_{\frac{\mu+\nu}{2}}(t)$

$$\doteqdot \frac{\Gamma(\lambda+2)\, p^{-\lambda-1}}{2^{\frac{\mu+\nu}{2}}\, \Gamma\left(\frac{\mu+\nu+2}{2}\right)}\, {}_2F_1\left(\begin{matrix}\frac{\lambda}{2}+1, \frac{\lambda+3}{2}; \\ \frac{\mu+\nu+2}{2};\end{matrix} -\frac{1}{p^2}\right) \equiv \psi(p)$$

and $x^{2\mu-\lambda} f(x) = x^{\frac{3\mu-\nu}{2}+1} J_{\frac{\mu+\nu}{2}}(x)$

$$\doteqdot \frac{\Gamma(2\mu+2)\, p^{-2\mu-1}}{2^{\frac{\mu+\nu}{2}}\, \Gamma\left(\frac{\mu+\nu+2}{2}\right)}\, {}_2F_1\left(\begin{matrix}\mu+1, \mu+\frac{3}{2}; \\ \frac{\mu+\nu+2}{2};\end{matrix} -\frac{1}{p^2}\right). \qquad (3.1)$$

Hence from (2.1) we get

$$\int_0^\infty {}_2F_1\left(\begin{matrix}\frac{\lambda}{2}+1, \frac{\lambda+3}{2}; \\ \frac{\mu+\nu+2}{2};\end{matrix} -\frac{1}{t^2}\right) {}_3F_2\left(\begin{matrix}\mu+\frac{3}{2}, \frac{\mu+\nu+3}{2}, \frac{\mu-\nu+3}{2}; \\ \frac{\lambda+3}{2}, \frac{\lambda}{2}+2;\end{matrix} t^2 p^2\right) dt$$

$$= \frac{\Gamma\left(\frac{\lambda+3}{2}\right)\Gamma\left(\frac{\lambda}{2}+2\right)\Gamma\left(\frac{\nu-\mu-1}{2}\right)\Gamma(2\mu+2)\, p^{-2\mu-3}}{2^{2\mu-\lambda+1}\,\Gamma\left(\mu+\frac{3}{2}\right)\Gamma\left(\frac{\mu+\nu+3}{2}\right)\Gamma(\lambda+2)}\, {}_2F_1\left(\begin{matrix}\mu+1, \mu+\frac{3}{2}; \\ \frac{\mu+\nu+2}{2};\end{matrix} -\frac{1}{p^2}\right).$$

In particular, if $\mu = \nu$, then (3.2)

$$\int_0^\infty {}_2F_1\left(\begin{matrix}\frac{\lambda}{2}+1, \frac{\lambda+3}{2}; \\ \nu+1;\end{matrix} -\frac{1}{t^2}\right) {}_3F_2\left(\begin{matrix}\nu+\frac{3}{2}, \nu+\frac{3}{2}, \frac{3}{2}; \\ \frac{\lambda+3}{2}, \frac{\lambda}{2}+2;\end{matrix} t^2 p^2\right) dt$$

$$= -\frac{\sqrt{\pi}\,\Gamma\left(\frac{\lambda+3}{2}\right)\Gamma\left(\frac{\lambda}{2}+2\right)\Gamma(2\nu+2)\, p^{-2\nu-3}}{2^{2\nu-\lambda}\,\Gamma\left(\nu+\frac{3}{2}\right)\Gamma\left(\frac{2\nu+3}{2}\right)\Gamma(\lambda+2)}\, {}_1F_0\left(\nu+\frac{3}{2}; -\frac{1}{p^2}\right).$$

4. Example 2. Let

$$t^{n+\mu+\frac{1}{2}} K_n(t) \quad \text{be} \quad R_{\mu,\nu}, \quad \text{where} \quad \nu = \mu + 2n$$

then $f(t) = t^{n+\lambda+1} K_n(t)$

$$\doteqdot \frac{\sin[(n+\lambda+1)\pi]\,\Gamma(\lambda+2)}{\sin[(2n+\lambda+1)\pi]\,(p^2-1)^{\frac{n+\lambda+2}{2}}}\, Q^n_{n+\lambda+1}\left(\frac{p}{\sqrt{p^2-1}}\right) \equiv \psi(p)$$

and

$$x^{2\mu-\lambda} f(x) = x^{2\mu+n+1} K_n(x)$$

$$\doteqdot \frac{\sin[(2\mu+n+1)\pi]\,\Gamma(2\mu+2)}{\sin[(2\mu+2n+1)\pi]\,(p^2-1)^{\frac{2\mu+n+2}{2}}}\, Q^n_{2\mu+n+1}\left(\frac{p}{\sqrt{p^2-1}}\right).$$

Hence from (2.1) we get

$$\int_0^\infty \frac{t^{\lambda+1}}{(t^2-1)^{\frac{2\mu+n+2}{2}}}\, Q^n_{2\mu+n+1}\left(\frac{t}{\sqrt{t^2-1}}\right) {}_3F_2\left(\begin{matrix}\mu+\frac{3}{2}, \frac{2\mu+2n+3}{2}, \frac{3-2n}{2}; \\ \frac{\lambda+3}{2}, \frac{\lambda}{2}+2;\end{matrix}\; t^2 p^2\right) dt$$

$$= \frac{\Gamma(2\mu+2)\,\Gamma\left(\frac{\lambda+3}{2}\right)\Gamma\left(\frac{\lambda}{2}+2\right)\Gamma\left(\frac{2n-1}{2}\right)\sin[(2\mu+n+1)\pi]\sin[(2n+\lambda+1)\pi]}{2^{2\mu-\lambda+1}\,\Gamma\left(\mu+\frac{3}{2}\right)\Gamma\left(\frac{2\mu+2n+3}{2}\right)\Gamma(\lambda+2)\sin[(2\mu+n+1)\pi]\sin[(n+\lambda+1)\pi]}$$

$$\cdot \frac{1}{p^2\,(p^2-1)^{\frac{2\mu+n+2}{2}}}\, Q^n_{2\mu+n+1}\left(\frac{p}{\sqrt{p^2-1}}\right), \quad R(2\mu+2n-\lambda) > 0. \quad (4.1)$$

I am thankful to Prof. S. C. Mitra and Dr. B. Singh for their kind help and encouragement in the preparation of this paper.

References

[1] *Rao, C. V. H.:* Messenger of Mathematics, Vol. 47 (1918), 134–137.
[2] *Watson, G. N.:* Quart. Jour. of Math., Vol. 2 (1931), 298–309.
[3] *Goldstein:* Proc. Lond. Math. Soc., Vol. 2 (1932), 34, P (103).
[4] *Erdélyi, A.*, and others: Tables of Integral Transforms, *i* and *ii*, Bateman Project.

Zusammenfassung

Der Autor beweist einen Satz aus dem Gebiet der Laplace-Transformationen und illustriert ihn an zwei Beispielen.

Résumé

Dans la présente note, l'auteur établit un théorème appartenant à la transformation de Laplace et en donne deux applications.

Riassunto

L'autore prova un teorema riguardante la trasformazione di Laplace e ne da due esempi.