

A theorem in operational calculus

Autor(en): **Dahiya, R.S.**

Objektyp: **Article**

Zeitschrift: **Mitteilungen / Vereinigung Schweizerischer
Versicherungsmathematiker = Bulletin / Association des Actuaire
Suisse = Bulletin / Association of Swiss Actuaries**

Band (Jahr): **64 (1964)**

PDF erstellt am: **22.07.2024**

Persistenter Link: <https://doi.org/10.5169/seals-966968>

Nutzungsbedingungen

Die ETH-Bibliothek ist Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Inhalten der Zeitschriften. Die Rechte liegen in der Regel bei den Herausgebern.

Die auf der Plattform e-periodica veröffentlichten Dokumente stehen für nicht-kommerzielle Zwecke in Lehre und Forschung sowie für die private Nutzung frei zur Verfügung. Einzelne Dateien oder Ausdrucke aus diesem Angebot können zusammen mit diesen Nutzungsbedingungen und den korrekten Herkunftsbezeichnungen weitergegeben werden.

Das Veröffentlichen von Bildern in Print- und Online-Publikationen ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Die systematische Speicherung von Teilen des elektronischen Angebots auf anderen Servern bedarf ebenfalls des schriftlichen Einverständnisses der Rechteinhaber.

Haftungsausschluss

Alle Angaben erfolgen ohne Gewähr für Vollständigkeit oder Richtigkeit. Es wird keine Haftung übernommen für Schäden durch die Verwendung von Informationen aus diesem Online-Angebot oder durch das Fehlen von Informationen. Dies gilt auch für Inhalte Dritter, die über dieses Angebot zugänglich sind.

A Theorem in Operational Calculus

By *R. S. Dahiya, Pilani (Rajasthan), India*

Summary

The author proves a theorem in the field of Laplace transformations. Two examples show the possible applications.

1. Introduction. The integral equation

$$\Phi(p) = p \int_0^{\infty} e^{-pt} f(t) dt, \quad R(p) > 0. \quad (1.1)$$

represents the classical Laplace transform and the functions $\Phi(p)$ and $f(t)$ related by (1.1), are said to be operationally related to each other. $\Phi(p)$ is called the image and $f(t)$ the original. Symbolically we can write

$$\Phi(p) \doteq f(t) \quad \text{or} \quad f(t) \doteq \Phi(p).$$

The transforms, we have dealt with, are either the Hankel transform or another transform in which the kernel is designated by $\tilde{\omega}_{\mu,\nu}(x)$, where

$$\tilde{\omega}_{\mu,\nu}(x) = \sqrt{x} \int_0^{\infty} J_{\nu}\left(\frac{x}{t}\right) J_{\mu}(t) \frac{dt}{t}. \quad (1.2)$$

The integral on the right was first evaluated by C.V.H. Rao [1] and that it plays the role of a transform was conjectured by Watson [2]. Later on Bhatnagar has proved in detail that it plays the role of a transform. In this paper, we make use of Goldstein's theorem [3] as follows:

Let (i) $\Phi_1(p) \doteq f_1(t)$,

(ii) $\Phi_2(p) \doteq f_2(t)$,

then after applying Goldstein's theorem, we get

$$\int_0^\infty f_1(t) \Phi_2(t) \frac{dt}{t} = \int_0^\infty \Phi_1(t) f_2(t) \frac{dt}{t},$$

where $\Phi(p)$ is the image and $f(t)$ is the original.

2. Theorem. Let

(i) $\psi(p) \doteq f(x)$,

(ii) $t^{\mu-\lambda-\frac{1}{2}} f(t)$ be $R_{\mu,\nu}$,

then

$$x^{2\mu-\lambda} f(x) \doteq \frac{2^{2\mu-\lambda+1} \Gamma\left(\mu + \frac{3}{2}\right) \Gamma\left(\frac{\mu + \nu + 3}{2}\right)}{\Gamma\left(\frac{\lambda + 3}{2}\right) \Gamma\left(\frac{\lambda}{2} + 2\right) \Gamma\left(\frac{\nu - \mu - 1}{2}\right)} \cdot p^2 \int_0^\infty t^{\lambda+1} \psi(t) {}_3F_2\left(\begin{matrix} \mu + \frac{3}{2}, \frac{\mu + \nu + 3}{2}, \frac{\nu - \mu + 3}{2}; \\ \frac{\lambda + 3}{2}, \frac{\lambda}{2} + 2; \end{matrix} t^2 p^2\right) dt, \quad (2.1)$$

provided $f(x)$ is absolutely integrable in $(0, \infty)$, $t^{\mu-\lambda-\frac{1}{2}} f(t)$ and $x^{2\mu-\lambda} f(x)$ are integrable in $(0, \infty)$, and $R(\mu) \geq -\frac{1}{2}$, $R(\nu) \geq -\frac{1}{2}$, $R(\lambda + 2) \leq 0$.

Proof: Let

$$p^m \tilde{\omega}_{\mu,\nu}(p) \doteq F(x)$$

then

$$x^\lambda F\left(\frac{1}{x}\right) \doteq p^{\frac{1-\lambda}{2}} \int_0^\infty J_{\lambda+1}(2\sqrt{xp}) x^{\frac{\lambda-1}{2}+m} \tilde{\omega}_{\mu,\nu}(x) dx, \quad (2.2)$$

$R(\lambda + 2) > 0$, $R(\lambda + m + \mu) \geq -\frac{3}{2}$, $R(\lambda + m + \nu) \geq -\frac{3}{2}$, $R(\lambda + 2m) < 0$

$$\text{or, } x^\lambda F\left(\frac{1}{x}\right) \doteq p^{\frac{1-\lambda}{2}} \int_0^\infty J_{\lambda+1}(2\sqrt{pt}) t^{\frac{\lambda-1}{2}+m} \tilde{\omega}_{\mu,\nu}(t) dt, \quad (2.3)$$

$$\text{or, } x^\lambda F\left(\frac{1}{x}\right) \doteq \frac{p^{1-m-\lambda}}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma(\lambda+m+s)}{\Gamma(-m-s+2)} 2^{1-2s} \frac{\Gamma\left(\frac{\mu-s}{2} + \frac{3}{4}\right) \Gamma\left(\frac{\nu-s}{2} + \frac{3}{4}\right)}{\Gamma\left(\frac{\mu+s}{2} + \frac{1}{4}\right) \Gamma\left(\frac{\nu+s}{2} + \frac{1}{4}\right)} p^{-s} ds, \quad (2.4)$$

$$\text{or, } x^\lambda F\left(\frac{1}{x}\right) \doteq \frac{p^{1-\lambda-m}}{2\pi i} \int_{c-i\infty}^{c+i\infty} 2^{\lambda+2m-1} \frac{\Gamma\left(\frac{\lambda+m+s}{2}\right) \Gamma\left(\frac{\lambda+m+s+1}{2}\right) \Gamma\left(\frac{\mu-s}{2} + \frac{3}{4}\right)}{\Gamma\left(\frac{-m-s+2}{2}\right) \Gamma\left(\frac{-m-s+3}{2}\right) \Gamma\left(\frac{\mu+s}{2} + \frac{1}{4}\right)} \cdot \frac{\Gamma\left(\frac{\nu-s}{2} + \frac{3}{4}\right)}{\Gamma\left(\frac{\nu+s}{2} + \frac{1}{4}\right)} p^{-s} ds. \quad (2.5)$$

By putting $m = \mu - \lambda + \frac{1}{2}$ and then evaluating the integral, we get

$$x^\lambda F\left(\frac{1}{x}\right) \doteq 2^{2\mu-\lambda+1} p^2 \sum_{n=0}^{\infty} \frac{(-1)^n}{(n)!} \frac{\Gamma\left(\mu+n+\frac{3}{2}\right) \Gamma\left(\frac{\mu+\nu+3}{2}+n\right)}{\Gamma\left(\frac{\lambda}{2}+n+\frac{3}{2}\right) \Gamma\left(\frac{\lambda}{2}+n+2\right) \Gamma\left(\frac{\nu-\mu}{2}-n-\frac{1}{2}\right)} p^{2n} \quad (2.6)$$

$$\text{or, } x^\lambda F\left(\frac{1}{x}\right) \doteq 2^{2\mu-\lambda+1} p^2 \frac{\Gamma\left(\mu+\frac{3}{2}\right) \Gamma\left(\frac{\mu+\nu+3}{2}\right)}{\Gamma\left(\frac{\lambda+3}{2}\right) \Gamma\left(\frac{\lambda}{2}+2\right) \Gamma\left(\frac{\nu-\mu-1}{2}\right)} \cdot {}_3F_2\left(\begin{matrix} \mu+\frac{3}{2}, \frac{\mu+\nu+3}{2}, \frac{\mu-\nu+3}{2}; \\ \frac{\lambda+3}{2}, \frac{\lambda}{2}+2; \end{matrix} p^2\right). \quad (2.7)$$

On writing $\frac{x}{t}$ for x we have

$$\left(\frac{x}{t}\right)^\lambda F\left(\frac{t}{x}\right) \doteq 2^{2\mu-\lambda+1} \frac{\Gamma\left(\mu + \frac{3}{2}\right) \Gamma\left(\frac{\mu + \nu + 3}{2}\right) p^2 t^2}{\Gamma\left(\frac{\lambda + 3}{2}\right) \Gamma\left(\frac{\lambda}{2} + 2\right) \Gamma\left(\frac{\nu - \mu - 1}{2}\right)} \cdot {}_3F_2\left(\begin{matrix} \mu + \frac{3}{2}, \frac{\mu + \nu + 3}{2}, \frac{\mu - \nu + 3}{2}; \\ \frac{\lambda + 3}{2}, \frac{\lambda}{2} + 2; \end{matrix} t^2 p^2\right). \quad (\text{A})$$

Let (i) $p^{\mu-\lambda+\frac{1}{2}} \tilde{\omega}_{\mu,\nu}(p) \doteq F(x), \quad (\text{B})$

(ii) $\psi(p) \doteq f(x), \quad (\text{C})$

Also from (B), $(ap)^{\mu-\lambda+\frac{1}{2}} \tilde{\omega}_{\mu,\nu}(ap) \doteq F\left(\frac{x}{a}\right). \quad (\text{D})$

Applying Goldstein's theorem to (C) and (D), we get

$$\int_0^\infty f(t) (at)^{\mu-\lambda+\frac{1}{2}} \tilde{\omega}_{\mu,\nu}(at) \frac{dt}{t} = \int_0^\infty \psi(t) F\left(\frac{t}{a}\right) \frac{dt}{t}. \quad (2.8)$$

On writing x for a and multiplying by x^λ to both sides, it follows

$$x^\lambda \int_0^\infty f(t) (xt)^{\mu-\lambda+\frac{1}{2}} \tilde{\omega}_{\mu,\nu}(xt) \frac{dt}{t} = \int_0^\infty \psi(t) F\left(\frac{t}{x}\right) \left(\frac{x}{t}\right)^\lambda t^{\lambda-1} dt. \quad (2.9)$$

Interpreting with the help of (A), we get

$$x^{\mu+\frac{1}{2}} \int_0^\infty f(t) t^{\mu-\lambda-\frac{1}{2}} \tilde{\omega}_{\mu,\nu}(xt) dt \doteq \frac{2^{2\mu-\lambda+1} \Gamma\left(\mu + \frac{3}{2}\right) \Gamma\left(\frac{\mu + \nu + 3}{2}\right)}{\Gamma\left(\frac{\lambda + 3}{2}\right) \Gamma\left(\frac{\lambda}{2} + 2\right) \Gamma\left(\frac{\nu - \mu - 1}{2}\right)} \cdot p^2 \int_0^\infty t^{\lambda+1} \psi(t) {}_3F_2\left(\begin{matrix} \mu + \frac{3}{2}, \frac{\mu + \nu + 3}{2}, \frac{\mu - \nu + 3}{2}; \\ \frac{\lambda + 3}{2}, \frac{\lambda}{2} + 2; \end{matrix} t^2 p^2\right) dt. \quad (2.10)$$

If $t^{\mu-\lambda-\frac{1}{2}} f(t)$ is $R_{\mu,\nu}$, we obtain the required result.

3. Example 1. Let $t^{\frac{\mu-\nu+1}{2}} J_{\frac{\mu+\nu}{2}}(t)$ be $R_{\mu,\nu}$,

then $f(t) = t^{\lambda - \frac{\nu+\mu}{2} + 1} J_{\frac{\mu+\nu}{2}}(t)$

$$\doteq \frac{\Gamma(\lambda+2) p^{-\lambda-1}}{2^{\frac{\mu+\nu}{2}} \Gamma\left(\frac{\mu+\nu+2}{2}\right)} {}_2F_1\left(\begin{matrix} \frac{\lambda}{2} + 1, \frac{\lambda+3}{2}; \\ \frac{\mu+\nu+2}{2}; \end{matrix} -\frac{1}{p^2}\right) \equiv \psi(p)$$

and $x^{2\mu-\lambda} f(x) = x^{\frac{3\mu-\nu}{2} + 1} J_{\frac{\mu+\nu}{2}}(x)$

$$\doteq \frac{\Gamma(2\mu+2) p^{-2\mu-1}}{2^{\frac{\mu+\nu}{2}} \Gamma\left(\frac{\mu+\nu+2}{2}\right)} {}_2F_1\left(\begin{matrix} \mu+1, \mu+\frac{3}{2}; \\ \frac{\mu+\nu+2}{2}; \end{matrix} -\frac{1}{p^2}\right). \quad (3.1)$$

Hence from (2.1) we get

$$\int_0^\infty {}_2F_1\left(\begin{matrix} \frac{\lambda}{2} + 1, \frac{\lambda+3}{2}; \\ \frac{\mu+\nu+2}{2}; \end{matrix} -\frac{1}{t^2}\right) {}_3F_2\left(\begin{matrix} \mu+\frac{3}{2}, \frac{\mu+\nu+3}{2}, \frac{\mu-\nu+3}{2}; \\ \frac{\lambda+3}{2}, \frac{\lambda}{2} + 2; \end{matrix} t^2 p^2\right) dt$$

$$\equiv \frac{\Gamma\left(\frac{\lambda+3}{2}\right) \Gamma\left(\frac{\lambda}{2} + 2\right) \Gamma\left(\frac{\nu-\mu-1}{2}\right) \Gamma(2\mu+2) p^{-2\mu-3}}{2^{2\mu-\lambda+1} \Gamma\left(\mu+\frac{3}{2}\right) \Gamma\left(\frac{\mu+\nu+3}{2}\right) \Gamma(\lambda+2)} {}_2F_1\left(\begin{matrix} \mu+1, \mu+\frac{3}{2}; \\ \frac{\mu+\nu+2}{2}; \end{matrix} -\frac{1}{p^2}\right).$$

(3.2)

In particular, if $\mu = \nu$, then

$$\int_0^\infty {}_2F_1\left(\begin{matrix} \frac{\lambda}{2} + 1, \frac{\lambda+3}{2}; \\ \nu+1; \end{matrix} -\frac{1}{t^2}\right) {}_3F_2\left(\begin{matrix} \nu+\frac{3}{2}, \nu+\frac{3}{2}, \frac{3}{2}; \\ \frac{\lambda+3}{2}, \frac{\lambda}{2} + 2; \end{matrix} t^2 p^2\right) dt$$

$$\equiv \frac{\sqrt{\pi} \Gamma\left(\frac{\lambda+3}{2}\right) \Gamma\left(\frac{\lambda}{2} + 2\right) \Gamma(2\nu+2) p^{-2\nu-3}}{2^{2\nu-\lambda} \Gamma\left(\nu+\frac{3}{2}\right) \Gamma\left(\frac{2\nu+3}{2}\right) \Gamma(\lambda+2)} {}_1F_0\left(\nu+\frac{3}{2}; -\frac{1}{p^2}\right).$$

4. Example 2. Let

$$t^{n+\mu+\frac{1}{2}} K_n(t) \quad \text{be} \quad R_{\mu,\nu}, \quad \text{where} \quad \nu = \mu + 2n$$

then $f(t) = t^{n+\lambda+1} K_n(t)$

$$\doteq \frac{\sin [(n + \lambda + 1) \pi] \Gamma(\lambda + 2)}{\sin [(2n + \lambda + 1) \pi] (p^2 - 1)^{\frac{n+\lambda+2}{2}}} Q_{n+\lambda+1}^n \left(\frac{p}{\sqrt{p^2 - 1}} \right) \equiv \psi(p)$$

and

$$x^{2\mu-\lambda} f(x) = x^{2\mu+n+1} K_n(x)$$

$$\doteq \frac{\sin [(2\mu + n + 1) \pi] \Gamma(2\mu + 2)}{\sin [(2\mu + 2n + 1) \pi] (p^2 - 1)^{\frac{2\mu+n+2}{2}}} Q_{2\mu+n+1}^n \left(\frac{p}{\sqrt{p^2 - 1}} \right).$$

Hence from (2.1) we get

$$\int_0^\infty \frac{t^{\lambda+1}}{(t^2-1)^{\frac{2\mu+n+2}{2}}} Q_{2\mu+n+1}^n \left(\frac{t}{\sqrt{t^2-1}} \right) {}_3F_2 \left(\begin{matrix} \mu + \frac{3}{2}, \frac{2\mu+2n+3}{2}, \frac{3-2n}{2}; \\ \frac{\lambda+3}{2}, \frac{\lambda}{2} + 2; \end{matrix} ; t^2 p^2 \right) dt$$

$$= \frac{\Gamma(2\mu+2) \Gamma\left(\frac{\lambda+3}{2}\right) \Gamma\left(\frac{\lambda}{2}+2\right) \Gamma\left(\frac{2n-1}{2}\right) \sin [(2\mu+n+1)\pi] \sin [(2n+\lambda+1)\pi]}{2^{2\mu-\lambda+1} \Gamma\left(\mu+\frac{3}{2}\right) \Gamma\left(\frac{2\mu+2n+3}{2}\right) \Gamma(\lambda+2) \sin [(2\mu+n+1)\pi] \sin [(n+\lambda+1)\pi]}$$

$$\cdot \frac{1}{p^2 (p^2-1)^{\frac{2\mu+n+2}{2}}} Q_{2\mu+n+1}^n \left(\frac{p}{\sqrt{p^2-1}} \right), \quad R(2\mu+2n-\lambda) > 0. \quad (4.1)$$

I am thankful to Prof. S.C. Mitra and Dr. B. Singh for their kind help and encouragement in the preparation of this paper.

References

- [1] Rao, C.V.H.: Messenger of Mathematics, Vol.47 (1918), 134–137.
- [2] Watson, G.N.: Quart. Jour. of Math., Vol.2 (1931), 298–309.
- [3] Goldstein: Proc. Lond. Math. Soc., Vol.2 (1932), 34, P (103).
- [4] Erdélyi, A., and others: Tables of Integral Transforms, *i* and *ii*, Bateman Project.

Zusammenfassung

Der Autor beweist einen Satz aus dem Gebiet der Laplace-Transformationen und illustriert ihn an zwei Beispielen.

Résumé

Dans la présente note, l'auteur établit un théorème appartenant à la transformation de Laplace et en donne deux applications.

Riassunto

L'autore prova un teorema riguardante la trasformazione di Laplace e ne dà due esempi.