A geometric proof of Borch's theorem

Autor(en): **Gerber, Hans U.**

Objekttyp: **Article**

Zeitschrift: **Mitteilungen / Vereinigung Schweizerischer Versicherungsmathematiker = Bulletin / Association des Actuaires Suisses = Bulletin / Association of Swiss Actuaries**

Band (Jahr): **75 (1975)**

PDF erstellt am: **22.07.2024**

Persistenter Link: https://doi.org/10.5169/seals-967114

A Geometric Proof of Borch's Theorem

By Hans U. Gerber

1. *Introduction.* Most of the existing proofs of Borch's Theorem are of an analytical nature (see [1], [2] p. 122–127, [3] p. 190–196, or [4]). The purpose of this note is to show how geometric arguments can be used to prove and interpret Borch's Theorem.

Let Y be a real random variable (defined in connection with an appropriate probability space). We interpret Y as the aggregate income of n insurance companies. To fix ideas, let us assume that Y has a bounded range. A *risk exchange* is a random vector $(Y_1, Y_2, \ldots, Y_n)$, for which

$$Y_1 + Y_2 + \ldots + Y_n = Y \quad \text{a.s.} \tag{1}$$

and for which the regularity condition (2) below is satisfied. Intuitively, Y_i is the share of company i after the exchange. We suppose that company i is primarily interested in $E[U_i(Y_i)]$, where $U_i(x)$, $-\infty < x < \infty$, is its utility function. We assume that the functions $U_1(x), \ldots, U_n(x)$ have the following properties:

(a) $U_i(x)$ is twice differentiable
(b) $U_i'(x) > 0$, $-\infty < x < \infty$
(c) $U_i''(x) < 0$, $-\infty < x < \infty$

$(i = 1, 2, \ldots, n)$. Condition (b) amounts to the trivial requirement that the utility of company i is an increasing function of its surplus. Condition (c) means that company i is a "risk averter". We are only interested in risk exchanges $(Y_1, \ldots, Y_n)$, for which

$$E[Y_i] \text{ and } E[U_i(Y_i)] \text{ exist} \tag{2}$$

for $i = 1, 2, \ldots, n$.

A risk exchange $(Y_i^*, \ldots, Y_n^*)$ is said to be *Pareto-optimal,* if for any risk exchange $(Y_1, \ldots, Y_n)$ the n simultaneous inequalities

$$E[U_i(Y_i)] \geq E[U_i(Y_i^*)] \tag{3}$$

are only possible if equality holds for $i = 1, 2, \ldots, n$. A sufficient condition for Pareto-optimality is the existence of positive constants $k_1, k_2, \ldots, k_n$, such that

$$\sum_{i=1}^{n} k_i\{E[U_i(Y_i)] - E[U_i(Y_i^*)]\} \leq 0 \tag{4}$$

for any risk exchange $(Y_1, \ldots, Y_n)$. We shall see that this condition is also necessary.

We observe that for a function $U(x)$ with properties (a) and (c) above the following inequality holds:

$$U(s) - U(t) \leq U'(t) \cdot (s-t) \tag{5}$$

for any pair of real numbers s, t (with equality holding only if $s = t$). This is best seen by plotting the graph of the function U (which is concave from below) and its tangent line at the point with coordinates $(t, U(t))$.

2. *The surface F_c*. For any real number c, let F_c denote the following surface in R^n:

$$F_c = \{(x_1, \ldots, x_n) | x_i = U_i(t_i), t_1 + \ldots + t_n = c\}. \tag{6}$$

In this context we call $(t_1, \ldots, t_n)$ the *coordinates* of the point $(x_1, \ldots, x_n)$. If $(x_1, \ldots, x_n) \in F_c$, $x_i = U_i(t_i)$, the normal vector at this point is parallel to the vector with components

$$U_1'(t_1)^{-1}, U_2'(t_2)^{-1}, \ldots, U_n'(t_n)^{-1}. \tag{7}$$

This can be seen as follows: Let $(y_1, \ldots, y_n)$ be another point on the surface F_c, say with coordinates $(s_1, \ldots, s_n)$. Formula (5) tells us that

$$y_i - x_i \leq U_i'(t_i) \cdot (s_i - t_i) \tag{8}$$

with equality holding only if $s_i = t_i$. Now we divide both sides of the above inequality by $U_i'(t_i)$. Since $s_1 + \ldots + s_n = t_1 + \ldots + t_n = c$, summation on both sides leads to the inequality

$$\sum_{i=1}^{n} U_i'(t_i)^{-1} \cdot (y_i - x_i) \leq 0, \tag{9}$$

with equality holding only if $(x_1, \ldots, x_n) = (y_1, \ldots, y_n)$. Thus the vector with components (7) is indeed parallel to the normal vector; moreover, formula (9) shows that the tangential plane at any point "supports" the surface in a strict sense. So F_c is an unbounded, convex surface in R^n.

Remark. The above considerations show that the normal vectors of F_c point to the first 2^n-tant. Also, for a given vector, there is at most one point in F_c at which this vector is the normal vector. Furthermore, the set of vectors for which there is a corresponding normal vector on F_c is independent of c. For example, if $U_i'(\infty) = 0$ for $i = 1, \dots, n$, or if $U_i'(-\infty) = \infty$ for $i = 1, \dots, n$, this set consists of the unit vectors of the open first 2^n-tant.

3. *Borch's Theorem.* Let $\kappa = (k_1, k_2, \dots, k_n)$ be a unit vector in the first 2^n-tant, such that for all c (or, equivalently, for at least one c) there is a point on F_c where κ is the normal vector. For such a κ we construct a risk exchange $(Y_2^\kappa, Y_2^\kappa, \dots, Y_n^\kappa)$ as follows: Let $(X_1^\kappa, X_2^\kappa, \dots, X_n^\kappa)$ denote the point on F_Y whose normal vector is κ. Then Y_i^κ is defined as the i-th coordinate of this point, $X_i^\kappa = U_i(Y_i^\kappa)$. The reader is invited to verify that $Y_1^\kappa, \dots, Y_n^\kappa$ is indeed a risk exchange.

Theorem. (a) Risk exchanges of the form $(Y_1^\kappa, \dots, Y_n^\kappa)$ are Pareto-optimal.
(b) If a risk exchange $(Y_1, \dots, Y_n)$ is not of this form, there is a κ such that $E[U_i(Y_i)] < E[U_i(Y_i^\kappa)]$ for $i = 1, 2, \dots, n$.

Proof: (a) Let $(Y_1, \dots, Y_n)$ be an arbitrary risk exchange, and let $(Y_1^\kappa, \dots, Y_n^\kappa)$ be one of the special form.
From inequality (9) we gather that

$$\sum_{i=1}^{n} k_i \cdot [U_i(Y_i) - U_i(Y_i^\kappa)] \le 0 \text{ a.s.} \tag{10}$$

Taking expected values, we obtain

$$\sum_{i=1}^{n} k_i \cdot \{E[U_i(Y_i)] - E[U_i(Y_i^\kappa)]\} \le 0, \tag{11}$$

which shows the Pareto-optimality of $(Y_1^\kappa, \dots, Y_n^\kappa)$. Furthermore, equality holds in (11) only if $Y_i = Y_i^\kappa$ a.s.
(b) Let us introduce the surface F and the solid S in R^n:

$$F = \{(x_1, \dots, x_n) | x_i = E[U_i(Y_i^\kappa)], i = 1, \dots, n\} \tag{12}$$

$$S = \{(x_1, \dots, x_n) | x_i = E[U_i(Y_i)], i = 1, \dots, n\}. \tag{13}$$

In the first definition the variation is extended over all risk exchanges of the special form, in the second over all risk exchanges. Inequality (11) shows that F is a convex surface (unbounded of course). Similarly, one can show that S is a convex solid whose boundary is F, where S is "below" F. From the last remark in the proof of part (a) it follows that the points of F can only be generated by risk exchanges of the special form q.e.d.

Remark. The reader may find it helpful to draw a picture in the case $n = 2$. Here S is a two dimensional region that is unbounded to the south and west. On the north-east side S is bounded by the curve F.

Hans U. Gerber
Department of Mathematics
The University of Michigan
Ann Arbor, MI 48104
U.S.A.

References

[1] *Borch, K.H.* (1960): The Safety Loading of Reinsurance Premiums. Skand. Akt. *43*, 163–184.
[2] *Borch, K.H.* (1974). The Mathematical Theory of Insurance, An Annotated Selection of Papers. Heath, Lexington.
[3] *Bühlmann, H.* (1970). Mathematical Methods in Risk Theory. Springer, Berlin.
[4] *Du Mouchel, W.H.* (1968): The Pareto-optimality of an n-company reinsurance treaty. Skand. Akt. *51*, 165–170.

Zusammenfassung

Der Satz von Borch wird bewiesen und interpretiert anhand von rein geometrischen Betrachtungen.

Résumé

Le théorème de Borch est démontré et interpreté par des considérations entièrement géométriques.

Riassunto

Il teorema di Borch è dimostrato ed interpretato di modo puramente geometrico.

Summary

The theorem of Borch is proved and interpreted by purely geometric arguments.