An invariance property of the Swiss premium calculation principle

Autor(en): **De Vylder, Floriaan / Goovaerts, Marc**

Objekttyp: **Article**

Zeitschrift: **Mitteilungen / Vereinigung Schweizerischer Versicherungsmathematiker = Bulletin / Association des Actuaires Suisses = Bulletin / Association of Swiss Actuaries**

Band (Jahr): **79 (1979)**

PDF erstellt am: **22.07.2024**

Persistenter Link: https://doi.org/10.5169/seals-967124

An invariance property of the Swiss premium calculation principle

by Floriaan De Vylder and Marc Goovaerts, Belgium

Abstract

The Swiss premium calculation principle, introduced in [1], associates to a given risk X a premium p, solution of the equation

$$Ef(X-zp)=f((1-z)p),$$

where $z \in [0, 1]$ and f is a continuous strictly monotonic function. The particular cases $z=0$ or $z=1$ are considered in detail in [2], [3], [6].
Let f be replaced by the strictly monotonic continuous function g. Then we prove that p does not change, for no X, iff g is a linear combination $g=\alpha+\beta f(\alpha, \beta$ constants). This result is a rather direct generalization of the one found in [6], Chapter III, in the case $z=0$.
As an application, we prove a result very recently indicated in [4], i.e. that the Swiss premium calculation principle is additive (that means that the premium corresponding to the sum of two independent risks is the sum of the premiums corresponding to each risk) iff f is linear or exponential.
In another illustration we show that the Swiss premium calculation principle is iterative (see section 7.) iff f is linear or exponential. The difficult part of this result was proved first in [5].
In further applications we characterize translation invariance, positive homogeneity, symmetry, homogeneity, multiplicativity.
The concepts are defined and the results proved for risks with arbitrary signs. From the practical point of view it may be preferable to have a theory restricted to nonnegative risks. That the results are also valid in the latter case, at least for $0<z<1$, is indicated in the last section.

1. Definition of the Swiss principle

Let f be a continuous strictly monotonic real function defined on $R=\,]-\infty, +\infty[$. Let $z \in [0, 1]$. Let X be a real random variable (we consider it as defined by the distribution of its probability mass on R). Let $p \in R$. Then we say that p is the *Swiss premium* associated to the risk X iff

$$Ef(X-zp)=f((1-z)p). \tag{1}$$

Generally, $p = p(X, f, z)$ depends on X, f, z. Practically X is nonnegative and p is positive, but there is no need to introduce these restrictions here. All numbers, functions, random variables considered in this note are finite.
We define:
$\mathfrak{D}_2$ = class of discrete random variables X with strictly positive probability masses in two distinct points (but no elsewhere).
$\mathfrak{B}$ = class of bounded random variables X (those with all probability mass in a finite interval).
The Swiss premium is uniquely defined for all $X \in \mathfrak{B}$, whatever be f, z satisfying the indicated conditions. Of course, it generally exists for X in a larger class, depending in fact on f and z. For simplicity, statements are made for $X \in \mathfrak{D}_2$ or $X \in \mathfrak{B}$ in the sequel, but often immediate extensions are evident.

2. Lemma

Let f, g be strictly monotonic. Then there exist $\alpha, \beta \in R$, $\beta \neq 0$ such that

$$g(x) = \alpha + \beta f(x), \qquad (x \in R), \tag{2}$$

iff

$$\frac{f(x'')-f(x)}{f(x'')-f(x')} = \frac{g(x'')-g(x)}{g(x'')-g(x')} \tag{3}$$

for each $x' < x < x''$.

Remarks

- We omit the trivial demonstration of this lemma.
- It is interesting, however, that no continuity assumptions are made on the involved functions f, g. Indeed, starting an argument with continuous functions, continuity may be lost through limiting procedures, e.g. simply by taking derivatives.
- In the preceding lemma, the assumptions make sure that the involved denominators are not zero. When one of the functions is not supposed to be strictly monotonic, the lemma has an immediate extension used in section 5.
- Next theorem in section 4 is most appealing, but practically, in this paper, it is always the version in section 5 that is used. Indeed, even when one starts an argument with very regular functions, a function g may appear in the discussion that is not strictly monotonic. For instance, the derivative of a linear

function is a constant and a constant cannot be used as function f in the defining relation (1). However, next relation (13) makes sense when g is a constant.

3. Lemma

Let $z \in [0, 1]$ and let f be continuous strictly monotonic. Let $X \in \mathfrak{D}_2$ have the distribution defined as follows:

$$P(X = a) = 1-t, \quad P(X = b) = t, \quad (a<b, 0<t<1). \tag{4}$$

Let $p(t) = p(X, f, z)$. Then $p(t)$ takes any value $c \in]a, b[$ when t varies in $]0, 1[$. Moreover, $p(t)$ is strictly increasing. This implies that $p(t)$ is continuous on $]0, 1[$. (It is easily seen that these results are also valid for the closed intervals and that $p(0) = a$, $p(1) = b$.)

Demonstration

Here the defining relation (1) for $p(t)$ becomes

$$(1-t) f(a-zp(t)) + t f(b-zp(t)) = f((1-z)p(t)). \tag{5}$$

Let $a<c<b$ and let us replace $p(t)$ by c in (5). Then it will be sufficient to show that (5) gives a corresponding t strictly between 0 and 1. We have

$$a-zc<(1-z)c<b-zc$$

and then

$$f(a-zc)<f((1-z)c)<f(b-zc),$$

because we may assume f strictly increasing. Since (5) can be written

$$\frac{1-t}{t} = \frac{f(b-zp(t))-f((1-z)p(t))}{f((1-z)p(t))-f(a-zp(t))} \tag{6}$$

where the last member is strictly positive, we have $0<t<1$. From (6) it is immediate that $p(t)$ is strictly increasing.

4. Theorem

Let f, g be continuous strictly monotonic and let $z \in [0, 1[$. Then

$$p(X, f, z) = p(X, g, z), \qquad (X \in \mathfrak{B}), \tag{7}$$

iff

$$g(x) = \alpha + \beta f(x) \qquad (x \in R) \tag{8}$$

for some $\alpha, \beta \in R$, $\beta \neq 0$. (The possible dependence of α, β on z is not indicated since z is fixed.)

Demonstration

Let f, g be connected by (8) and let p satisfy

$$E\,g(X - zp) = g((1-z)\,p). \tag{9}$$

Then (1) follows immediatly, i.e. (7) is true.
Conversely, let (7) hold. In order to prove (8) it is sufficient to show that (3) holds. Let $x' < x < x''$. We define

$$a = x' + \frac{zx}{1-z}, \qquad b = x'' + \frac{zx}{1-z}, \qquad c = \frac{x}{1-z}. \tag{10}$$

Then $a < c < b$. Let us consider the random variable X defined in the lemma of 3. By that lemma there is a $t \in]0, 1[$ such that the premium $p(t)$ (it amounts to the same to calculate it with f or with g) satisfies

$$p(t) = c = \frac{x}{1-z}.$$

Then the relations (10) become:

$$x = (1-z)\,p(t), \qquad x' = a - z\,p(t), \qquad x'' = b - z\,p(t)$$

and (6) becomes:

$$\frac{1-t}{t} = \frac{f(x'') - f(x)}{f(x) - f(x')}. \tag{11}$$

Since we have the same relation, with f replaced by g, (3) follows.

5. Extension

In next theorem no assumptions are made on g, but the conclusion does not say that $\beta \neq 0$.

Theorem

Let $z \in [0, 1[$, let f be continuous strictly monotonic and let[1] g be any function. Suppose that for each $X \in \mathfrak{D}_2$, the root p of

$$Ef(X - zp) = f((1-z)p) \tag{12}$$

is also a root of

$$E g(X - zp) = g((1-z)p). \tag{13}$$

Then

$$g(x) = \alpha + \beta f(x) \qquad (x \in R) \tag{14}$$

for some $\alpha, \beta \in R$.

Demonstration

Under these assumptions, one part of the lemma in 2. is valid in the following way:
If

$$(f(x'') - f(x))(g(x'') - g(x')) = (f(x'') - f(x'))(g(x'') - g(x)) \tag{15}$$

for any $x' < x < x''$, then (2) remains valid, but maybe $\beta = 0$.

Then the demonstration goes on as in the preceding theorem. The relation (11) can be written

$$(1-t)(f(x) - f(x')) = t(f(x'') - f(x)).$$

From the assumptions follows that a similar relation can be written down but with f replaced by g. Then (15) follows.

[1] Note that whatever be the function g, not necessarily Lebesgue measurable, $E\, g\,(X - zp)$ makes sense for $X \in \mathfrak{D}_2$.

6. Application: Additivity

For fixed f, z and variable X, the *premium* $p(X, f, z)$ is said to be *additive* iff

$$p(X + Y, f, z) = p(X, f, z) + p(Y, f, z),$$

whatever be the independent random variables X, Y in $\mathfrak{B}$.

Theorem

Let $f' > o$ exist. Let $z \in [0, 1[$. Then $p(X, f, z)$ is additive iff

$$f(x) = \alpha + \beta e^{ax} \quad \text{or} \quad f(x) = \alpha + \beta x, \ (x \in R) \tag{16}$$

for some a, α, $\beta \in R$.

Demonstration

If f is linear or exponential, then clearly, $p(X, f, z)$ is additive.
Conversely, let $p(X, f, z)$ be additive. Let $X \in \mathfrak{D}_2$ and let $p = p(X, f, z)$ satisfy (1). Let Y be the random variable with its total probability mass 1 placed in the point $c \neq o$. Then $c = p(Y, f, z)$ and X, Y are independent. By the additivity assumption $p + c$ is solution of

$$Ef(X + c - zp - zc) = f((1-z)(p + c)). \tag{17}$$

Substracting (1), then dividing by $(1-z)c$ and letting $c \to o$, we obtain: $Ef'(X - zp) = f'((1-z)p)$. Since $X \in \mathfrak{D}_2$, the limiting procedure is evidently permitted. By the preceding theorem we have $f' = \alpha + \beta f$. Then f' is continuous, since f is, and classical calculus applies. If $\beta = o$, then f is linear. If $\beta \neq o$ then f is exponential.
Remark. It is not necessary to suppose that f' exists, in the preceding theorem. Indeed, let us assume only that f is continuous strictly monotonic. Let g be defined by the relation

$$g(x) = f(x + c(1-z)), \quad (x \in R).$$

Then (17) reads

$$E\,g(X - zp) = g((1-z)p)$$

and by the theorem in 5., g is a linear function of f. But now the coefficients may depend on c (not assumed non-zero here). Replacing $c(1-z)$ by y, it is seen that there exist functions $\alpha(y)$, $\beta(y)$ such that

$$f(x + y) = \alpha(y) + \beta(y) f(x), \quad (x, y \in R). \tag{18}$$

It can probably be shown by elementary arguments that this relation implies that f is linear or exponential. Here we shall use the preceding theorem and the fact that f has a derivative (finite) in at least one point x_0. (By Lebesgue measure theory it is known that it has one a.e.).
Let $z \in R$. Define $y = z - x_0$. Then, by (18), for $\Delta z \neq o$:

$$\frac{1}{\Delta z}(f(z + \Delta z) - f(z)) = \frac{1}{\Delta z}(f(x_0 + \Delta z + y) - f(x_0 + y))$$

$$= \beta(y)\frac{1}{\Delta z}(f(x_0 + \Delta z) - f(x_0)).$$

For $\Delta z \to o$, the last expression has a finite limit. This means that f is derivable in z, i.e. in any point.

7. Application: Iterativity

For fixed f and z, the Swiss premium calculation principle is said to be *iterative* iff for each X, $Y \in \mathfrak{B}$, the relations

$$E(f(X - zq(Y))/Y) = f((1-z)q(Y)) \tag{19}$$

$$Ef(q(Y) - zp) = f((1-z)p) \tag{20}$$

imply

$$Ef(X - zp) = f((1-z)p). \tag{21}$$

Thus, in (19), the principle is applied conditionally, given Y. The obtained premium is a random variable $q(Y)$, a function of the conditioning variable Y. Then (20) defines the premium p corresponding to $q(Y)$. Iterativity means that the premium corresponding directly to X is the same as that one corresponding to $q(Y)$.

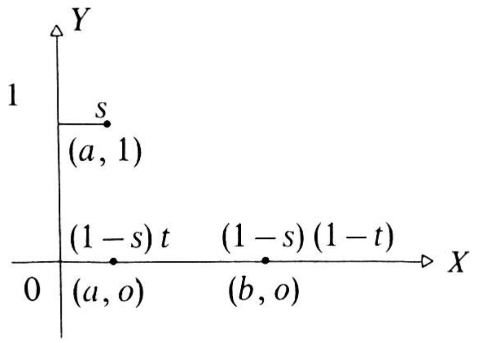
Theorem

Let $f' > o$ exist. Let $z \in]0, 1[$. Then the corresponding Swiss premium calculation principle is iterative iff f is linear or exponential (i.e. iff (16) holds).

Demonstration

If f is linear or exponential, then the principle is iterative. This easily follows from general properties of conditional expectations, in particular the property: $E(Xg(Y)/Y) = g(Y)E(X/Y)$.

Conversely, let the principle be iterative. Let us consider the couple (X, Y) with distribution defined by (see fig. 1):



(fig. 1)

$$P(X = a, Y = o) = (1-s)t, \; P(X = b, Y = o) = (1-s)(1-t), \quad P(X = a, Y = 1) = s, \; (a<b, o<s<1, o<t<1). \tag{22}$$

Then, for the conditional variable $X_0 = X_{/Y=o}$:

$$P(X_0 = a) = t, \; P(X_0 = b) = 1-t.$$

For the conditional variable $X_1 = X_{/Y=1}$: $P(X_1 = a) = 1$.
For the marginal variable X : $P(X = a) = s+t-st$, $P(X = b) = (1-s)(1-t)$.
By (19), the variable $q(Y)$ is distributed as follows:

$$P(q(Y) = a) = s, \; P(q(Y) = q) = (1-s)t + (1-s)(1-t) = 1-s,$$

where q is solution of

$$tf(a-zq) + (1-t)f(b-zq) = f((1-z)q). \tag{23}$$

Then (20), (21) become

$$sf(a-zp) + (1-s)f(q-zp) = f((1-z)p), \tag{24}$$

$$(t+s-st)f(a-zp) + (1-s)(1-t)f(b-zp) = f((1-z)p). \tag{25}$$

From (24), (25) results:

$$tf(a-zp) + (1-t)f(b-zp) = f(q-zp). \tag{26}$$

Now we consider s as variable. From (23) results that q does not depend on s. From (23), (25) and the lemma in 3. results that $p \neq q$. The same lemma and (25) show that p is a continuous function of s and that $\lim p = q$ when $s \to o$. Therefore, taking the difference of (23) and (26), then dividing by $z(q-p)$ and letting $s \to o$, one obtains

$$t f'(a-zq) + (1-t) f'(b-zq) = f'((1-z)q). \tag{27}$$

But (23), (27) can be written down as

$$Ef(X_0 - zq) = f((1-z)q), \qquad Ef'(X_0 - zq) = f'((1-z)q).$$

Since X_0 can be considered as being an arbitrary variable in $\mathfrak{D}_2$, one has $f' = \alpha + \beta f$ by the theorem in 5. Then f is linear or exponential.

8. Application: Translation invariance

For fixed f, z, the Swiss premium calculation principle is said to be *translation invariant* iff for each $X \in \mathfrak{B}$, $c \in R$ (or $c > o$; alternative definition),

$$p(X + c, f, z) = c + p(X, f, z). \tag{28}$$

Since for any constant random variable $Y = c$, we have $c = p(Y, f, c)$ and since such a random variable is independent from any other random variable, translation invariance seems to be a much less stringent condition than additivity. However, the demonstration of the theorem in 6. shows that the concepts are equivalent when applied to the Swiss premium calculation principle.

Theorem

Let $f' > o$ exist and let $z \in [0, 1[$. Then the corresponding Swiss premium calculation principle is translation invariant iff it is additive, i.e. iff f is linear or exponential (i.e. iff (16) holds).

9. Application: Homogeneity

For fixed f, z, the Swiss premium calculation principle is said to be *homogeneous* iff for each $X \in \mathfrak{B}$, $c \in R$,

$$p(cX, f, z) = c\, p(X, f, z). \tag{29}$$

It is said to be *positively homogeneous* iff (29) holds for each $X \in \mathfrak{B}$ and $c > o$. It is said to be *symmetric* iff (29) holds for each $X \in \mathfrak{B}$ and $c = -1$.

Theorem

Let $z \in [0, 1[$ and let f be continuous strictly monotonic. Then the corresponding Swiss premium calculation principle is positively homogeneous iff

$$f(x) = \begin{cases} \alpha + \beta x^r & (x \geq o) \\ \alpha - \gamma(-x)^r & (x < o) \end{cases} \tag{30}$$

for some $\alpha, \beta, \gamma \in R$, $\beta\gamma > o$, $r > o$.

Demonstration.

Let f be defined by (30) and let $X \in \mathfrak{B}$. Let F be the distribution function of X. Then the premium p corresponding to X satisfies

$$-\gamma \int_{-\infty}^{zp} (zp - x)^r \, dF(x) + \beta \int_{zp}^{\infty} (x - zp)^r \, dF(x) = \delta(1-z)^r |p|^r, \tag{31}$$

where $\delta = -\gamma$ if $p < o$ and $\delta = \beta$ if $p \geq o$. Multiplying (31) by c^r, where $c > o$, one obtains, after evident transformations, the relation expressing that cp is the premium corresponding to cX.

Conversely, let the principle be positively homogeneous. We may assume that f is strictly increasing. Substracting $f(o)$ from f, we may also assume that $f(o) = o$. Let $X \in \mathfrak{D}_2$ and let p be the premium corresponding to X. Then (1) holds and by the homogeneity assumption: $Ef(cX - zcp) = f((1-z)cp)$, $(c > o)$. This relation can be written down as $E\,g(X - zp) = g((1-z)p)$, where g is defined by $g(x) = f(cx)$, $(x \in R)$. By the theorem in 5. g is a linear combination of f. The coefficients may depend on c. Thus, there exist functions $\alpha(c)$, $\beta(c)$ such that

$$f(cx) = \alpha(c) + \beta(c) f(x), \; (x \in R, \; c > o).$$

For $x = o$, this relation shows that $\alpha(c) = o$. Then

$$f(cx) = \beta(c) f(x), \; (x \in R, \; c > o). \tag{32}$$

For $x = 1$, we find that $\beta(c) = f(c)/f(1)$ and then (32) becomes

$$f(cx) = f(c) f(x)/f(1), \; (x \in R, \; c > o). \tag{33}$$

We now consider two cases.

Case 1: $x \geq o$. We define $f_1(x) = f(x)/f(1)$. Then (33) gives: $f_1(cx) = f_1(c) f_1(x)$, $(x \geq o, c > o)$. Since f_1 is continuous this implies that $f_1(x) = x^r$ for some $r \in R$.

Since $f_1(o) = o$, we must have $r > o$. Then $f(x) = \beta x^r$ where $\beta = f(1)$.
Case 2: $x < o$. Then we put $y = -x$ and we define $g(y) = -f(y) = -f(-x)$. Then $y > o$, $g(y) > o$ and (32) implies: $g(cy) = \beta(c)\, g(y)$. Similarly as in case 1 we have $g(y) = \gamma y^s$, where $s > o$, $\gamma = g(1) = -f(-1)$. Then $f(x) = -\gamma(-x)^s$, $(x < o)$.
Using the obtained results in (33), we see that we must have $r = s$.

Theorem

Let $z \in [0, 1[$ and let f be continuous strictly monotonic. Then the corresponding Swiss premium calculation principle is symmetric iff

$$f(x) = \alpha + \beta f_0(x), \qquad (x \in R), \tag{34}$$

where α, $\beta \in R$ and f_0 is a function satisfying $f_0(-x) = -f_0(x)$, $(x \in R)$. (This means that f_0 is an "odd function".)

Demonstration

For f defined by (34), the principle is clearly symmetric.
Conversely, let the principle be symmetric. Let $f_0(x) = f(x) - f(o)$, $(x \in R)$. For the premium p corresponding to $X \in \mathfrak{D}_2$, we have $E f_0(X - zp) = f_0((1-z)\,p)$ and then, by the symmetry assumption, $E f_0(-X + zp) = f_0(-(1-z)\,p)$. Let $g_0(x) = f_0(-x)$. Then $E\, g_0(X - zp) = g_0((1-z)\,p)$. By the theorem in 5. we have $g_0 = \alpha_0 + \beta_0 f_0$. Letting $x = o$, we see that $\alpha_0 = o$. Then $f_0(-x) = \beta_0 f_0(x)$ $= \beta_0^2 f_0(-x)$. Then $\beta_0^2 = 1$ and obviously, $\beta_0 = -1$.

Theorem

Let z $[0, 1[$ and let f be continuous strictly monotonic. Then the corresponding Swiss premium calculation principle is homogeneous iff

$$f(x) = \alpha + \beta |x|^r \operatorname{sign} x, \qquad (x \in R),$$

for some α, $\beta \in R$, $r > o$.

Demonstration

This results from the two preceding theorems since the principle is homogeneous iff it is positively homogeneous and symmetric.

10. Application: Multiplicativity

For fixed f, z, the Swiss premium calculation principle is said to be *multiplicative* iff

$$p(XY, f, z) = p(X, f, z)\, p(Y, f, z) \tag{35}$$

for each couple of independent random variables X, $Y \in \mathfrak{B}$.
In the next demonstration, we could use the last theorem of the preceding section. We prefer to use the first one because we later indicate versions of the theorems using only nonnegative risks and in that case the following demonstration remains valid.

Theorem

Let $z \in \,]0, 1[$ and let f be continuous strictly monotonic. Then the corresponding Swiss premium calculation principle is multiplicative iff

$$f(x) = \alpha + \beta x, \qquad (x \in R), \tag{36}$$

for some α, $\beta \in R$.

Demonstration

For f given by (36), the principle is clearly multiplicative.
Conversely, let the principle be multiplicative. Then it is positively homogeneous and (30), where we may assume $\alpha = o$, holds. We consider the independent random variables X, Y distributed as follows

$$P(X = o) = 1 - s,\ P(X = 1) = s,\ (o < s < 1),$$

$$P(Y = o) = 1 - t,\ P(Y = 1) = t,\ (o < t < 1).$$

Then

$$P(XY = o) = 1 - st,\ P(XY = 1) = st.$$

Let $p(s)$, $p(t)$ be the premiums corresponding to X, Y respectively. Then, by the lemma in 3., $p(s) > o$, $p(t) > o$. By the multiplicativity assumption, $p(s)\,p(t)$ is the premium corresponding to XY.
By (30), we have:

$$-(1-t)\gamma z^r p^r(t) \qquad + t\,\beta(1 - z\,p(t))^r \qquad = \beta(1-z)^r p^r(t),$$

$$-(1-st)\gamma z^r p^r(s)\,p^r(t) + st\,\beta(1 - z\,p(s)\,p(t))^r = \beta(1-z)^r p^r(s)\,p^r(t).$$

Multiplying the first relation by $p^r(s)$ and substracting the last, then dividing by t, we obtain

$$-(1-s)\,\gamma\, z^r\, p^r(s)\, p^r(t) + s\,\beta(1-z\,p(s)\,p(t))^r = \beta(1-z\,p(t))^r\, p^r(s). \quad (37)$$

For $t \to o$, we have $p(t) \to o$ by the lemma in 3. and then the last relation gives: $s = p^r(s)$. Now we replace $p(s)$ by x and note that x can take any value in $]0, 1[$ by the lemma in 3. We divide by $s = p^r(s)$ and replace $p(t)$ by 1. This is permitted since for $t \to 1$, we have $p(t) \to 1$ by the lemma in 3. Then it results from (37) that

$$-\gamma(1-x^r)\,z^r + \beta(1-zx)^r = \beta(1-z)^r, \ (o<x<1). \quad (38)$$

We may take the derivative in x:

$$\gamma\, x^{r-1}\, z^r = \beta\, z(1-zx)^{r-1}, \ (o<x<1).$$

Letting $x \to o$, we see that the last relation is only possible if $r = 1$. Then (38) becomes

$$-\gamma z + \gamma x z - \beta x z = -\beta z, \ (o<x<1).$$

For $x \to o$, we obtain $\gamma = \beta$.

11. Special cases ($z = o$ or $z = 1$)

Previous arguments break down for $z = o$ or for $z = 1$, but some results remain valid.

The theorems in 4. and 5. are valid for $z = 1$. We sketch a proof. One works with the functions $f_0(x) = f(x) - f(o)$, $g_0(x) = g(x) - g(o)$ having the property: $f_0(o) = o$, $g_0(o) = o$. The lemma in 2. is replaced by the following simpler one: If $f_0(x)\, g_0(x') = f_0(x')\, g_0(x)$ for each $x < o < x'$, then $g_0 = \alpha f_0$ and conversely. One makes use of the variable X defined in the lemma of 3. and relation (6) that now becomes

$$\frac{1-t}{t} = -\frac{f_0(b-p(t))}{f_0(a-p(t))} \quad (39)$$

(or that relation written without denominators). Then the point is to show that, given $x < o < x'$, one can find a, b, t such that $a - p(t) = x$, $b - p(t) = x'$. This is immediate since one can take $a = x$, $b = x'$, $p(t) = o$ and then for t the one resulting from (39).

These extensions can be used to generalize previous results. But we must insist on the fact that there are situations where the cases $z = o$ or $z = 1$ are far from trivial. See e.g. [2], [3], [6].

12. Only nonnegative risks

From a practical point of view, it may be preferable to consider only nonnegative risks. For $z \in]0, 1[$, almost all preceding results can be seen to be valid in that situation. Then the theorem in section 5 must be replaced by the following theorem, the proof of which is left to the reader.

Theorem

Let $z \in]0, 1[$. Let f be continuous strictly monotonic and let g be any function. Suppose that for all nonnegative $X \in \mathfrak{D}_2$, the root p of $Ef(X-zp) = f((1-z)p)$ is also a root of $E\,g(X-zp) = g((1-z)p)$. Then $g(x) = \alpha + \beta f(x)$, $(x \in R)$ for some $\alpha, \beta \in R$.

Appendix.
When is the Swiss premium calculation principle expectation exceeding?

1. Let f be continuous strictly increasing, $z \in [0, 1]$. Let $X \in \mathfrak{B}$ be such that $P(a \leq X \leq b) = 1$. We define

$$g_1(x) = Ef(X-zx), \qquad g_2(x) = f((1-z)x).$$

Then g_1, g_2 are continuous, g_1 is decreasing, g_2 is increasing and at least one of the functions g_1, g_2 is strictly monotonic.
Since $a \leq X \leq b$ (a.s.): $f(a-zx) \leq f(X-zx) \leq f(b-zx)$ (a.s.)
Taking expectations: $f(a-zx) \leq Ef(X-zx) \leq f(b-zx)$.
For $x = a$ the first inequality gives $g_2(a) \leq g_1(a)$ and for $x = b$ the second gives $g_1(b) \leq g_2(b)$. This implies that the Swiss premium $p = p(X, f, z)$, the root of $g_1(x) = g_2(x)$, exists and is unique and moreover that $a \leq p(X, f, z) \leq b$.
2. In [1] it is proved that when f is convex, the stronger result $E(X) \leq p(X, f, z) \leq b$ holds (see 4. below). From the practical point of view, the property $E(X) \leq p(X, f, z)$ is essential, at least for positive risks X.
3. For fixed f, z, the Swiss premium calculation principle will be called *expectation exceeding* iff $E(X) \leq p(X, f, z)$ for each nonnegative $X \in \mathfrak{B}$.
In this appendix we show that it is not essential, for that property to hold, that f be convex on the whole of R. When $z = o$, this is immediate, since then f is not used on $]-\infty, o[$ if $X \geq o$, but we shall give a less trivial example.

4. For fixed f, z, the Swiss premium calculation principle is expectation exceeding iff for each nonnegative $X \in \mathfrak{B}$.

$$f((1-z)m) \leq Ef(X-zm), \tag{A}$$

where $m = EX$. This is immediate from the discussion in 1. (Draw the graphs representing g_1, g_2.) Note that when f is convex, Jensen's inequality gives $f(E(X-zm)) \leq Ef(X-zm)$ and that this is exactly relation (A).

5. For $f(x) = x^3$ $(x \in R)$, relation (A) becomes

$$(1-z)^3 m^3 \leq E(X-zm)^3,$$
$$(1-3z+3z^2-z^3)m^3 \leq E(X^3-3X^2zm+3Xz^2m^2-z^3m^3),$$
$$(1-3z)m^3 \leq EX^3 - 3zmEX^2.$$

For $z = 1/3$:

$$EX \cdot EX^2 \leq EX^3. \tag{B}$$

6. It is easy to see that (B) holds for each nonnegative $X \in \mathfrak{B}$. Indeed, let F be the distribution function of X. Then

$$2(EX^3 - EX \cdot EX^2) = \iint (x^3+y^3-xy^2-x^2y)\,dF(x)\,dF(y)$$
$$= \iint (x-y)(x^2-y^2)\,dF(x)\,dF(y) \geq o,$$

where the inequality holds because $(x-y)(x^2-y^2) \geq o$ when $x, y \geq o$.

Conclusion of the Appendix

For $f(x) = x^3$ $(x \in R)$, $z = 1/3$, the Swiss premium calculation principle is expectation exceeding, although f is not convex on R.

It would be most interesting to have a simple characterization, in terms of f and z only, of the property of being expectation exceeding. The problem seems to be intricate. It is easily seen that it is not sufficient that f be convex on $[0, \infty[$.

The discussion in this appendix justifies the general definition of the Swiss premium calculation principle that we adopted at the start. Of course, it is easier to draw practical conclusions when f is supposed to be convex, but a lot of work can be done without that assumption.

References

[1] *Bühlmann H., Gagliardi B., Gerber H., Straub E.* (1977): Some inequalities for stop-loss premiums. ASTIN-Bulletin, Vol. IX, p. 75–83.
[2] *Gerber H.* (1974): On additive premium calculation principles. ASTIN-Bulletin, Vol. VII, p. 215–222.
[3] *Gerber H.* (1974): On iterative premium calculation principles. MVSV. Heft 2, p. 163–172.
[4] *Goovaerts M., de Vylder F.* (to appear): A note on additive premium calculation principles.
[5] *Goovaerts M., de Vylder F.* (to appear): A note on iterative premium calculation principles.
[6] *Hardy G., Littlewood J., Polya G.* (1959): Inequalities. Cambridge University Press.

Floriaan De Vylder
Professeur
à l'Université de Louvain
315, Steenweg of Ninove
B-9400 Denderleeuw

Marc Goovaerts
Instituut voor Actuariele
Wetenschappen
Katholieke Universiteit te Leuven
Dekenstraat 2
B-3000 Leuven

Zusammenfassung

Das sogenannte «Schweizer» Prämienberechnungsprinzip wird in verschiedener Hinsicht verallgemeinert und kommentiert.

Résumé

Le soi-disant «principe suisse» de tarification est analysé et généralisé sous plusieurs aspects et commenté.

Riassunto

Viene analizzato e generalizzato da diversi punti di vista il cosidetto principio svizzero del calcolo di premi.

Summary

The Swiss premium calculation principle is analyzed and generalized in various directions.