D. Kurzmitteilungen

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Some comments to D. Zagorac: Ein Beitrag zur Intervallschätzung der Glaubwürdigkeitsparameter

(Mitteilungen 1/1981, 67–75)

1. Referring to the Fisher Lemma Zagorac states that

$$\frac{(nN-1)W}{\sigma^2+\sigma_0^2}=\sum_i\sum_j\frac{(X_{ij}-m_0)^2}{\sigma^2+\sigma_0^2}-nN\frac{(M-m_0)^2}{\sigma^2+\sigma_0^2}$$

is χ^2-distributed with $nN-1$ degrees of freedom. However, in the present case one has to be a bit careful, as for fixed j the X_{ij}'s are dependent when $\sigma_0^2 \neq 0$. By use of orthogonal transformations it can be shown that

$$Q_0=\sum_{j=1}^{N}\sum_{i=1}^{n}(X_{ij}-X_{.j})^2,$$

$$Q=n\sum_{j=1}^{N}(X_{.j}-M)^2,$$

and M are independent, and that Q_0/σ^2 and $Q/(\sigma^2+n\sigma_0^2)$ are χ^2-distributed with $N(n-1)$ resp. $N-1$ degrees of freedom.
From this we see that

$$\frac{(nN-1)W}{\sigma^2+\sigma_0^2}=\frac{Q_0+Q}{\sigma^2+\sigma_0^2}$$

is χ^2-distributed with $nN-1$ degrees of freedom if and only if $\sigma_0^2=0$.

2. Referring to the above, the confidence regions proposed in Zagorac's (1981) Sections 7.1, 7.2, and 7.4 seem questionable, and in the rest of this note I shall propose other confidence regions.

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3. Confidence interval for m_0.
As $X_{.1}, \ldots, X_{.N}$ are independent and identically normally distributed with mean m_0,

$$T = \frac{M - m_0}{\sqrt{Q}} \sqrt{nN(N-1)}$$

is Student distributed with $N-1$ degrees of freedom. From this follows that

$$J_1 = \left[M - t_{N-1,1-\frac{\epsilon}{2}} \sqrt{\frac{Q}{nN(N-1)}}, M + t_{N-1,1-\frac{\epsilon}{2}} \sqrt{\frac{Q}{nN(N-1)}} \right],$$

with $t_{N-1,1-\frac{\epsilon}{2}}$ being the $1-\frac{\epsilon}{2}$ fractile of the Student distribution with $N-1$ degrees of freedom, is a $1-\epsilon$ confidence interval for m_0.

4. Confidence interval for σ_0^2.
Let $\chi^2_{v,\alpha}$ denote the α fractile in the χ^2-distribution with v degrees of freedom. We have

$$Pr\left(\frac{Q}{\sigma^2 + n\sigma_0^2} \geqq \chi^2_{N-1,\epsilon}\right) = 1 - \epsilon,$$

that is,

$$Pr\left(\frac{Q}{n\chi^2_{N-1,\epsilon}} \geqq \frac{\sigma^2}{n} + \sigma_0^2\right) = 1 - \epsilon.$$

Then

$$Pr\left(\frac{Q}{n\chi^2_{N-1,\epsilon}} \geqq \sigma_0^2\right) > 1 - \epsilon,$$

and from this follows that

$$J_2 = \left(0, \frac{Q}{n\chi^2_{N-1,\epsilon}}\right]$$

is a $1-\epsilon$ confidence interval for σ_0^2.

5. Confidence interval for σ^2.
Zagorac uses that Q_0/σ^2 is χ^2-distributed with $N(n-1)$ degrees of freedom and gets as a $1-\epsilon$ confidence interval for σ^2

$$I_3 = \left[\frac{N(n-1)\,V}{q_1}, \frac{N(n-1)\,V}{q_2}\right],$$

where q_1 and q_2 are determined such that

$$Pr\left(q_2 \leqq \frac{Q_0}{\sigma^2} \leqq q_1\right) = 1 - \epsilon. \tag{1}$$

Zagorac uses $q_1 = \chi^2_{N(n-1),1-\frac{\epsilon}{2}}$ and $q_2 = \chi^2_{N(n-1),\frac{\epsilon}{2}}$.

Referring to Sverdrup (1967, Sections XIII 3.2 and XIII 4.2) I would recommend q_1 and q_2 determined by (1) and

$$\frac{\log q_1 - \log q_2}{q_1 - q_2} = \frac{1}{N(n-1)};$$

this will make the confidence interval unbiased.

6. Confidence interval for $\kappa = \sigma^2/\sigma_0^2$.
The credibility estimator of m_j is

$$\tilde{m}_j = \frac{n}{n+\kappa} X_{.j} + \frac{\kappa}{n+\kappa} m_0.$$

We see that in this formula σ^2 and σ_0^2 appear only through κ. Hence, it may be interesting to construct a confidence interval for κ.

For this purpose we use that $(1 + n\kappa^{-1})^{-1} F$ with

$$F = \frac{Q}{Q_0} \frac{N(n-1)}{N-1}$$

is Fisher distributed with $N-1$ and $N(n-1)$ degrees of freedom. Let f_1 and f_2 be determined such that

$$Pr\,(f_1 \leqq (1 + n\kappa^{-1})^{-1} F \leqq f_2) = 1 - \epsilon. \tag{2}$$

Then

$$Pr\left(\frac{n}{\frac{F}{f_1} - 1} \leqq \kappa \leqq \frac{n}{\frac{F}{f_2} - 1}\right) = 1 - \epsilon,$$

and

$$J_3 = \left[\frac{n}{\frac{F}{f_1} - 1}, \frac{n}{\frac{F}{f_2} - 1} \right]$$

is a $1 - \epsilon$ confidence interval for κ.

A simple choice of f_1 and f_2 would be the $\frac{\epsilon}{2}$ resp. $1 - \frac{\epsilon}{2}$ fractile of the Fisher distribution with $N-1$ and $N(n-1)$ degrees of freedom. However, to get the confidence interval unbiased, I suggest to find f_1 and f_2 from the equations (2) and

$$\frac{\log\left(1 + f_2 \frac{N-1}{N(n-1)}\right) - \log\left(1 + f_1 \frac{N-1}{N(n-1)}\right)}{\log f_2 - \log f_1} = \frac{N-1}{Nn-1}.$$

7. Confidence regions for (m_0, κ) and $(m_0, \sigma^2, \sigma_0^2)$.
From the Bonfferoni inequality we get

$$Pr((m_0 \in J_1) \cap (\kappa \in J_3)) \geqq 1 - 2\epsilon,$$

and thus

$$A_1 = J_1 \times J_3$$

is a $1 - 2\epsilon$ confidence region for (m_0, κ).
By using the independence of M, Q, and Q_0 we may construct a confidence region for $(m_0, \sigma^2, \sigma_0^2)$ without using inequalities like the Bonfferoni one.
Let $g_{1-\frac{\epsilon}{2}}$ denote the $1 - \frac{\epsilon}{2}$ fractile of the normal distribution $N(0, 1)$. Then, as M has the distribution $N\left(m_0, \sqrt{\frac{1}{N}\left(\frac{\sigma^2}{n} + \sigma_0^2\right)}\right)$,

$$Pr\left(M - g_{1-\frac{\epsilon}{2}} \sqrt{\frac{\sigma^2 + n\sigma_0^2}{nN}} \leqq m_0 \leqq M + g_{1-\frac{\epsilon}{2}} \sqrt{\frac{\sigma^2 + n\sigma_0^2}{nN}}\right) = 1 - \epsilon,$$

and for given $\sigma^2 + n\sigma_0^2$

$$J_4(\sigma^2 + n\sigma_0^2) = \left[M - g_{1-\frac{\epsilon}{2}} \sqrt{\frac{\sigma^2 + n\sigma_0^2}{nN}}, M + g_{1-\frac{\epsilon}{2}} \sqrt{\frac{\sigma^2 + n\sigma_0^2}{nN}} \right]$$

is a $1 - \epsilon$ confidence interval for m_0.

Using that $Q/(\sigma^2+n\sigma_0^2)$ is χ^2-distributed with $N-1$ degrees of freedom, we get, analogously to Section 5, that

$$J_5=\left[\frac{Q}{p_1},\frac{Q}{p_2}\right],$$

with p_1 and p_2 satisfying

$$Pr\left(p_2\leqq\frac{Q}{\sigma^2+n\sigma_0^2}\leqq p_1\right)=1-\in$$

is a $1-\in$ confidence interval for $\sigma^2+n\sigma_0^2$.

From the independence of M, Q, and Q_0 we now get

$$Pr((m_0\in J_4(\sigma^2+n\sigma_0^2))\cap(\sigma^2+n\sigma_0^2\in J_5)\cap(\sigma^2\in I_3))=$$
$$Pr(m_0\in J_4(\sigma^2+n\sigma_0^2))\,Pr(\sigma^2+n\sigma_0^2\in J_5)\,Pr(\sigma^2\in I_3)=$$
$$(1-\in)^3,$$

and

$$A_2=\{(x,y,z):x\in J_4(y+nz),z+ny\in J_5,z\in I_3\}$$

is a $(1-\in)^3$ confidence region for $(m_0,\sigma_0^2,\sigma^2)$.

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