Numerical probabilities of ruin when expected claim numbers are large

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Introduction

Numerical probabilities of eventual ruin, i.e. during a period when the number of expected claims is very large (theoretically infinite), can be traced back to Lundberg himself (1926, Kap. 8). Using his newly derived asymptotic formula for the probability of eventual ruin based on a large initial risk reserve (free surplus) of x_0, namely αe^{-Rx_0} (Lundberg, 1926, p. 26), and with four different claim size distributions: (i) all claims unity, (ii) exponentially distributed claims, (iii) gamma with index 2 claims, and (iv) an actual life insurance distribution of sums at risk from De Förenade, Lundberg's own company, he evaluated the proper fraction α for five different R-values, namely $0.05j$, $j=1,2,4,6,10$, showing that it decreased considerably over this range.
An exact value of the probability of eventual ruin when claims are occurring in time as a Poisson process (Lundberg's invariable assumption) and the claim size distribution is exponential, (ii) above, was published by Cramér (1930) as

$$\psi(w)=\frac{1}{1+\eta}\,e^{-\eta w/(1+\eta)}$$

where the monetary unit is the mean claim, η is the risk loading on the unit premium per expected claim, and w is the number of average claims that the company has available as initial risk reserve. The value of $\psi(w)$ for the uniform claim distribution, (i), had already been given by Erlang (1909) in a telephone delay problem (see Seal, 1969, 4.12). These formulas would have permitted the practitioner to evaluate exact eventual ruin probabilities at will for the two specified claim distributions.
The next published numerical probability of eventual ruin occurs in Cramér (1955, p. 45). It was based on a distribution of fire insurance claims graduated to produce the sum of an exponential and a truncated Pareto $v=1.75$ density, namely

$$b(y)=Ae^{-\alpha y}+B(y+6)^{-2.75}\qquad 0<y<500$$

where $b(\cdot)$ is written for the claim density, $A=4.897954$, $B=4.503$ and $\alpha=5.514588$. The asymptotic value of $\psi(w)$ was computed by Lundberg's formula already mentioned, the value of R being 0.00736. The «exact» four-decimal values of $\psi(w)$ were stated to have been calculated by Cramér's integral equation, (1) hereafter, and Nils Wikstad tells me that research among some old papers suggests that this computation was made using trapezoidal quadrature. The asymptotic formula is rather poor, not even being correct in the second decimal place, but the largest illustrative value of w is only 100.

At the time Cramér wrote, Ammeter's (1948) suggestion to replace risk theory's Poisson process of successive claims in a portfolio of policies by the negative binomial (Pólya) process had not yet produced statistical confirmation other than his own single example (however remember O. Lundberg's (1940) fittings of this process to successive sicknesses of individual insured), but the next twenty-five years were to see substantial evidence that this new process, as generalized by Thyrion (1959) to become the mixed Poisson process, was widely applicable in nonlife insurance (Seal, 1969, pp. 15–28; Brichler, 1971; Thyrion, 1972). In fact no other claim number process has yet been shown to agree with actual statistics of claim occurrences. Types of distribution of independent claim sizes are just as limited, for apart from the Pareto and lognormal distributions, namely with distribution functions, respectively,

$$B(y)=1-\left(1+\frac{y}{v-1}\right)^{-v} \qquad 0<y<\infty$$

and

$$B(y)=\Phi\left(\frac{\ln y-\xi}{\sigma}\right) \qquad 0<y<\infty$$

where $\Phi(\cdot)$ is the distribution function of Normal (0, 1), we are not aware that any has been fitted successfully to actual claim sizes in actuarial history (Seal, 1969, pp. 29–31; Benckert & Jung, 1974). For example, of the four published fittings of mixed exponential claims distribution none was tested statistically for goodness of fit and in two of the articles the authors themselves expressed doubts about the adequacy of the results.

There are few published calculations of $\psi(w)$, other than its upper bound, since Cramér's (1955) example. First were those of Grandell & Segerdahl (1971) and Bohman (1971). In both cases the gamma distribution was chosen for claim sizes, and while the joint authors utilized Lundberg's asymptotic formula Bohman illustrated the numerical inversion of the characteristic function (Laplace transform) of $\psi(w)$. Both sets of results agreed, essentially to five decimal places, with calculations by an «exact» formula of Thorin (1973). Then came Thorin & Wikstad's (1973) calculations including those for a Pareto claim distribution, and their further calculations in (1977) for lognormal claims assuming a Poisson process of claim occurrences.

This brief historical review has been necessary to show that the numerical calculation of $\psi(w)$, or as we will write it $1-U(w)$ to emphasize that $U(\cdot)$ can be regarded as a distribution function (Beekman, 1969), has rarely been attempted with a practical distribution of nonlife claim sizes. Furthermore, the calculation of $\psi(w)$ has never been made with a mixed Poisson, in particular a negative binomial, claims occurrence process with infinitely large expected claims.

We intend to try to fill the lacuna left by prior published work, namely to produce numerical values of $\psi(w)$ based on Poisson claim occurrences and Pareto or lognormal claim sizes. The formulas specifically used for this purpose, among others, by Thorin & Wikstad (1973, 1977) would naturally take pride of place among our procedures and, in fact, Nils Wikstad has kindly furnished me with a FORTRAN program to calculate $\psi(w)$. But those who have studied the two papers cited may think that the general technique developed there, using what Feller (1971, XI. 7) calls "deep complex variable methods", is rather involved for the calculation of the *relatively* straightforward ruin probabilities we require. After all, Cramér's integral equation, (1) hereafter, provides the "exact" results for any Poisson process of claim occurrences. And the final paragraphs of this paper show how these can usually be employed for mixed Poisson claim occurrence processes. This is why we have persisted in trying to develop simple numerical methods available to every practitioner.

The Cramér integral equation for Poisson claim occurrences

Lundberg's (1909) partial differential equation for $\chi(u,v)$, the probability density for a (first) ruin amounting to v given an initial risk reserve of u, was transformed by Cramér (1926) into an integral equation of Volterra type, namely

$$\int_0^\infty \chi(u,v)\,dv = \psi(u) = \frac{1}{1+\eta}\int_u^\infty \{1-B(v)\}\,dv + \frac{1}{1+\eta}\int_0^u \psi(u-v)\,\{1-B(v)\}\,dv$$

where $\psi(u)$ is the probability of eventual ruin with an initial risk reserve of u times the average claim and a continuously paid risk premium of $1+\eta$, and $B(\cdot)$ is the distribution function of individual claim amounts with conventional unit mean.

Claims are supposed to be occurring according to a Poisson process with a unit expected interval between successive claims. Cramér did not publish a proof of this relation until (1930).
We write $U(w) \equiv 1-\psi(w)$ and Cramér's relation becomes, $w \geq 0$,

$$U(w)=\frac{\eta}{1+\eta}+\frac{1}{1+\eta}\int_0^w U(w-y)\,\{1-B(y)\}\,dy$$

$$\equiv U(0)+\int_0^w U(w-y)h(y)\,dy \qquad (1)$$

This is proved by Feller (1971, VI. 5 and XI. 7) with his $R(\cdot) \equiv U(\cdot)$.
We have mentioned that Cramér's (1955, p. 45) "exact" method of solution of (1) involved the use of trapezoidal quadrature. However quadrature by repeated Simpson suggested by Seal (1978, p. 59) has never, we believe, been extended to large w-values and the necessity of small increments in w results in a computer run of three minutes or more. Nevertheless this was an obvious first trial in the numerical production of $U(w)$ from (1).
The results appeared to be satisfactory. For lognormal $(-\frac{1}{2}, 1)$ (the unit σ being approximately the value deduced from observations by Bühlmann & Hartmann (1956) and ζ then equals $-\frac{1}{2}$ for the mean claim to be unity) with a step in w equal to 0.1 the $U(w)$ values shown in Table A were obtained, and for $w \leq 20$ there was three decimal agreement with values produced using a w-step of 0.01. For Pareto $v=2.5$ (a "modern" value of v; Seal, 1980) the step of 0.1 achieved four decimal agreement at $U(20)$ with a 0.01 step, and a "check" of $U(250)$ based on evaluation of the right hand side of (1) using repeated Simpson with 0.1 increments reproduced the uncurtailed Table A value to six places of decimals.
We now proceed with other methods of calculating $U(w)$ that have been proposed either specifically or in similar situations.

Asymptotic formulas for large w

In Feller (1971, XI. 7) it is shown that it follows from (1) that

$$\lim_{w\to\infty}\{1-U(w)\}=\frac{\eta}{\int_0^\infty xe^{\kappa x}dB(x)-(1+\eta)}\,e^{-\kappa w} \qquad (2)$$

if the integral exists, where κ is obtained from

$$\int_0^\infty e^{\kappa y} dB(y) = 1 + (1+\eta)\kappa$$

This is the Lundberg (1926, p. 26) asymptotic formula already mentioned. The trouble with asymptotic formulas is that one needs some "exact" results to indicate from what value of the variable, w in this case, the asymptotic results can be used. However this problem does not arise when $B(y)$ is Pareto-v or lognormal (ζ, σ); the integral used in the determination of κ, conveniently by expanding the exponential and producing moments of $B(\cdot)$, diverges even though the moments of the lognormal are all finite. Those of the Pareto are infinite except for the mean and variance ($v > 2$). To meet this situation Thorin (1974) proposed a different asymptotic formula for Pareto-v and in a further article (Thorin & Wikstad, 1977) he extended his procedure to lognormal (ζ, σ) provided $\ln w > \zeta + \sigma^2$. We have called these formulas "Thorin asymptotic" in Table A and although the Pareto values are better than the lognormal neither set is acceptable.
The inapplicability of (2) for both Pareto and lognormal $B(\cdot)$ can be overcome by truncating these distributions at a remote tail abscissa. We experimented with tail abscissas y_t corresponding to $B(y_t) = 0.99995$ and found κ equal to 0.037819 and 0.065041, respectively. The corresponding asymptotic $U(w)$-values are shown in Table A. Those for the lognormal are in reasonable agreement with the quadrature of equation (1) but this cannot be said for Pareto.

Bartholomew's approximation

Cramér's relation (1) is a special case of the renewal equation. In the latter $U(0)$ becomes a function $g(w)$ and $h(y)$ is a density instead of proportional to the complement of a distribution function. Bartholomew (1963) proposed a very simple approximate solution to the equation where $g(w) = h(w)$ and provides several numerical examples in his 1973 book. In the case of equation (1) Bartholomew's formula becomes

$$U(w) \simeq U(0) \left(1 + \frac{wH(w)}{w - \int_0^w H(x)\,dx} \right) \tag{3}$$

where

$$H(x) = \int_0^x h(y)\,dy.$$

$H(x)$ and its integral can be evaluated for Pareto and lognormal $h(y)$ and numerical results are provided in Table A. We conclude that relation (3) does not constitute a good approximate solution to Cramér's equation.

The Laplace transform of $U(w)$

Another method of calculating $U(w)$ found in the literature is the inversion of the Laplace transform of $U(w)$. The latter is

$$\Upsilon(s) = \int_0^\infty e^{-sw} U(w)\,dw = \frac{\eta}{(1+\eta)s - 1 + \beta(s)} \tag{4}$$

where

$$\beta(s) = \int_0^\infty e^{-sy} b(y)\,dy$$

(Feller, 1971, XIV. 2(b); Cramér, 1930, (78)). Unfortunately $\beta(s)$ has no closed form when $B(\cdot)$ is lognormal but when the claims size distribution is Pareto-ν Seal (1980).

$$\beta(s) = \nu \int_0^\infty e^{-sby}(1+y)^{-\nu-1}\,dy = \nu e^{sb} E_{\nu+1}(sb), \qquad b = \nu - 1, \qquad Re(s) \geq 0$$

Seal (1980).

The numerical inversion of $\Upsilon(s)$ for $w = 50(50)450$ with $\beta(s)$ as above was effected by means of 24 term Gaussian integration described in the computer program GETBRM of Seal (1978, p. 83). The whole set required barely 10 seconds of CPU time but involved the writing of a program to produce $\beta(s)$, $s = c + iu$, in double precision arithmetic. The results are shown in Table A and are in exact three decimal agreement with the repeated Simpson quadrature of the integral in (1). This is gratifying because it indicates that straightforward, if long, quadrature of (1) provides correct results even for large w-values.

The moments of $U(w)$

It is quite easy to use (1) to evaluate numerically and exactly

$$\int_0^\infty w^n dU(w) \equiv m_n \qquad n=0,1,2,3,\ldots$$

the successive moments of the distribution function $U(w)$, in terms of the moments (about zero) p_j, $j=1,2,3,\ldots$, of the density $b(\cdot)$. For the lognormal (ζ,σ) all moments exist and are given by

$$p_j = e^{j\zeta + \frac{1}{2}j^2\sigma^2} \qquad \text{(Aitchison \& Brown, 1957, p. 8)}$$

but for Pareto-v it is seen (e.g. Seal, 1969, p. 30) that p_j is infinite when $j \geq v$. It can be shown that

$$m_n = \frac{p_{n+1}}{(n+1)\eta} + \frac{1}{\eta}\sum_{j=1}^{n-1}\frac{\eta^{(j-1)}}{j!} m_j p_{n-j+1}, \qquad m_0 = \frac{1}{1+\eta}, \qquad m_1 = \frac{p_2}{2\eta} \tag{5}$$

and this relation provides a recursion formula for m_n, $n=1,2,3,\ldots$. Note that $U(w)$, given by (1), is a monotonically increasing continuous function of w with a unique discontinuity at $w=0$. Any frequency distribution obtained from the above moments will thus have an aggregate area of $1-1/(1+\eta)$.
A family of curves based on the first four central moments of a distribution is that of Karl Pearson. The appropriate member of the family in any given case is determined from the position of (β_1, β_2) in the plane. Here

$$\beta_1 = \mu_3^2/\mu_2^3 \qquad \text{and} \qquad \beta_2 = \mu_4/\mu_2^2$$

where

$$\mu_n = \int_{-\infty}^{\infty} \{x-\mu\}^n p(x)\,dx$$

$p(\cdot)$ being the density whose moments are calculated and μ its mean.
For the lognormal $(-\frac{1}{2}, 1)$ use of (5) produces

$$\beta_1 = 4.813 \qquad \text{and} \qquad \beta_2 = 10.349$$

and

$$2\beta_2 - 3\beta_1 - 6 = 0.257$$

The smallness of this quantity is a criterion for Pearson's Type III (Elderton & Johnson, 1969, p. 78) which is a gamma distribution with a non-zero start, namely

$$p(x)=\frac{1}{a}\,\frac{p^{p+1}}{e^{p}\Gamma(p+1)}\left(1+\frac{x}{a}\right)^{\gamma a}e^{-\gamma x}$$

with index

$$p=\gamma a=\frac{4}{\beta_1}-1=-0.1690$$

and

$$a=\mu_2\gamma-1/\gamma=-59.936$$

In terms of the incomplete gamma ratio

$$\int_{-a}^{y} p(x)=P(p+1,\overline{a+y}\cdot\gamma)$$

Three-decimal values of

$$\frac{\eta}{1+\eta}+\left(1-\frac{\eta}{1+\eta}\right)\mathrm{P}(p+1,\overline{a+w}\cdot\gamma),\qquad \eta=0.1,\qquad w=25(25)150$$

are given in Table A. For $w\geq 75$ the results are in good agreement with quadrature using a step of 0.1 in w. We mention that a rather different incomplete gamma ratio approximation to $U(w)$ is given in Beekman (1969). With an unlimited number of moments m_n available no doubt some series approximation could be found to represent $U(w)$ pretty accurately.

Other solutions of the renewal equation

Bartholomew's (1963) approximate solution of the renewal equation was the culmination of a long series of attempts to produce numerical solutions based on different forms of what we have written as $h(\cdot)$. Some of these are described in Saxer (1958, Kap. 4) and the originator of the equation has detailed his preferred method in Lotka (1940). Writing the renewal form of equation as

$$u(x)=g(x)+\int_{0}^{x} u(x-y)f(y)\,dy$$

and subject to certain restrictions on the three functions involved (Feller, 1941) the solution takes the form

$$u(x) = \sum_{k=1}^{\infty} A_k e^{s_k x}$$

where the s_k are the roots of the equation

$$\phi(s) = \int_0^{\infty} e^{-sx} f(x)\, dx = 1$$

only s_1 being real, the remainder conjugate complex with real parts less than s_1. In Lotka (1940) the author uses the first four cumulants of $f(x)$, which is a Pearson Type I (beta) density, and finds the first 21 roots of $\phi(s) - 1$ by a method described there in detail. His last root is $s_{21} = -1.400 + 3.219i$ the real part of which is small enough to produce a vanishing term in the solution for $u(x)$. We followed Lotka's procedure with the first four "cumulants" of $h(x)$ based on lognormal $(-\frac{1}{2}, 1)$ but found that s_{32} was $-0.390 - 0.995i$, a value considered insufficiently small in view of the length of the series for $u(x) = U(w)$.

Conclusions for Poisson claim occurrences

Only Pareto and lognormal claim size distributions have been shown to be acceptable statistically and we have confined our attention to these. Asymptotic formulas for the probability of ruin with large w, the initial risk reserve expressed as a multiple of the mean claim amount, whether based on:

(a) Lundberg's 1926 formula;

(b) The same formula with 0.00005 of the largest claims eliminated; or

(c) Thorin's (1974, 1977) formulas for Pareto and lognormal claims size distributions;

have been shown to be inapplicable or to produce unreliable results. A simple formula for $U(w)$ developed for a slightly more general case by Bartholomew (1963, 1973) and used, we believe, fairly widely in industrial severance investigations fails quite badly in the actuarial field. On the other hand, repeated Simpson quadrature of the integral in Cramér's equation (1) (Seal, 1978, p. 59), although very long with small increments in w, produced good results which were confirmed by Gaussian inversion of the Laplace transform of $U(w)$ in the Pareto case. For the lognormal a fitting of a Pearson curve to the first four moments of $U(w)$ produced reasonably good results for the larger, important w-values.

Table A
Values of $U(w)$ with $\eta = 0.1$ by Various Methods

Pareto $v = 2.5$

w	Quadrature $\Delta w = 0.1$	Bartholomew (3)	Thorin asymptotic	Truncated Lundberg asymptotic	Gauss-24 inversion of $\Upsilon(s)$
50	0.836	0.695	0.950	0.881	0.836
100	0.948	0.809	0.982	0.982	0.948
150	0.978	0.861	0.990	0.997	0.978
200	0.988	0.890	0.994	1.000	0.988
250	0.993	0.909	0.995		0.993
300		0.922	0.996		0.995
350		0.932	0.997		0.996
400		0.940	0.998		0.997
450		0.946	0.998		0.998

Lognormal $(-\frac{1}{2}, 1)$

w	Quadrature $\Delta w = 0.1$	Bartholomew (3)	Thorin asymptotic	Truncated Lundberg asymptotic	Pearson Type III by moments
25	0.826	0.681	0.990	0.833	0.789
50	0.963	0.806	0.999	0.967	0.954
75	0.992	0.861	1.000	0.994	0.990
100	0.998	0.891		0.999	0.998
125	1.000	0.911		1.000	0.999
150		0.924			1.000
175		0.934			
200		0.942			
225		0.948			

Mixed Poisson claim occurrences

Writing $p_n(t)$ for the probability of n claims in an interval of time $(0, t)$ the mixed (or compound) Poisson process is defined by (O. Lundberg, 1940, p. 72)

$$p_n(t) = \int_0^\infty e^{-\lambda t} \frac{(\lambda t)^n}{n!} dM(\lambda) \tag{6}$$

where $M(\cdot)$ is a distribution function over the positive axis and is called the mixing distribution. It was introduced to actuaries by Dubourdieu (1938), without supporting observational evidence, as suitable for the probabilistic representation of automobile accidents, and Seal (1969, pp. 15–17) reports five statistical articles published before 1963 in which this model was applied. All these articles interpreted (6) as expressing the probability of n claims occurring to a *single* driver subject to a simple Poisson process of claim epochs, the claim intensity at any such epoch being λ chosen at random from a distribution function of λ's, namely $M(\lambda)$. However, as is implicit from O. Lundberg (1940, Ch. V), when a *portfolio* of two or more automobile accident policies is considered (6) means that:

(i) the claim intensity at any epoch in the interval $(0, t)$ depends on the aggregate number of accidents occurred prior to that epoch, and

(ii) given that an aggregate of n accidents has occurred in $(0, t)$ the epochs of those accidents are distributed uniformly with density t^{-n}.

These two properties of (6) are proved explicitly by Cane (1977).

The foregoing numerical investigation has been based on formulas derived with a Poisson point process of claim occurrences in mind. In Ammeter's (1948) original extension of the Poisson to negative binomial claim processes he showed how to transform the *aggregate* claim-size distribution by epoch t stemming from negative binomial claim input to one based on a Poisson input with the distribution of individual claim sizes modified to depend on the parameter t. This extension of risk theory based on Poisson claim numbers to negative binomial claim occurrences was generalized to any mixed Poisson claim process by Thyrion in a series of papers that culminated in that of 1969. An important result of Thyrion (1969) is that when the claim process is negative binomial, namely when

$$M'(\lambda)=\frac{h^h}{\Gamma(\eta)}\, e^{-h\lambda}\lambda^{h-1} \qquad h\geq 0, \qquad 0\leq\lambda<\infty$$

and (6) becomes

$$p_n(t)=\binom{n+h-1}{n}\left(\frac{h}{h+t}\right)^h\left(\frac{t}{h+t}\right)^n \qquad n=0,1,2,\ldots \qquad h>0$$

the mean of the adjusted individual claim size distribution is infinite when $t\to\infty$ and thus relation (2) cannot be used*. This is also true of the Hofmann (1955)

* Thyrion (1969) distinguishes between the case of a single infinite span and that of an infinity of independent spans of fixed length t (Ammeter's original case). The equation to determine κ of (2) for the latter model requires $B(y)$ to be changed to $\bar{B}(y, t)$ and κ on the right hand side to be multiplied by $\bar{p}_1$ as defined hereafter (Ammeter, 1948). In (1) $B(y)$ becomes $\bar{B}(y, t)$.

claim process

$$p_0(t)=\exp\left[\frac{b}{c(1-a)}\{1-(1+ct)^{1-a}\}\right] \qquad a>0$$

$$p_n(t)=(-1)^n\frac{t^n}{n!}p_0^{(n)}(t) \qquad n=1,2,3,\ldots$$

but not of the Double Poisson. These three are the only members of the mixed Poisson family that have been fitted successfully to actual distributions of the number of claims occurring in a fixed period t.
Thyrion (1969) writes the mixed Poisson claims process in the form

$$p_0(t)=e^{\theta(t)}, \quad \theta(0)=0, \quad (-1)^n\theta^{(n)}(t)\geq 0 \qquad n=1,2,\ldots$$

$$p_n(t)=(-1)^n\frac{t^n}{n!}p_0^{(n)}(t) \qquad n=1,2,3,\ldots$$

which is equivalent to (6). The adjusted claims size density is then given by

$$\bar{b}(x,t)=\sum_{m=1}^{\infty}(-1)^m\frac{t^m}{m!}\frac{\theta^{(m)}(t)}{-\theta(t)}b^{m^*}(x)$$

which, for negative binomial claims, takes the form

$$\bar{b}(x,t)=\frac{1}{\ln\left(1+\frac{t}{h}\right)}\sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{t}{h+t}\right)^m b^{m^*}(x)$$

in agreement with Ammeter (1948).
This density has mean

$$\bar{p}_1=\int_0^{\infty}x\bar{b}(x,t)\,dx=\sum_{m=1}^{\infty}(-1)^m\frac{t^m}{m!}\frac{\theta^{(m)}(t)}{-\theta(t)}\int_0^{\infty}xb^{m^*}(x)\,dx$$

$$=\frac{tp_1}{\theta(t)}\sum_{m=1}^{\infty}(-1)^{m-1}\frac{t^{m-1}}{(m-1)!}\theta^{(m)}(t)=\frac{tp_1}{\theta(t)}\theta'(t-t) \qquad \text{by Taylor's theorem}$$

$$=p_1\theta'(0)\frac{t}{\theta(t)} \qquad \text{or} \qquad p_1\frac{\frac{t}{h}}{\ln\left(1+\frac{t}{h}\right)} \qquad \text{for the negative binomial case.}$$

Now the probability of no claims in an infinite period of time being assumed to be zero, $\lim\limits_{t\to\infty}\theta(t)=-\infty$ and $\lim\limits_{t\to\infty}t/\theta(t)=\lim\limits_{t\to\infty}1/\theta'(t)$. The latter expression must be finite if $\bar{p}_1$ is to be finite as $t\to\infty$. It is easily shown that this requirement is not satisfied for the negative binomial, where $\theta'(t) = -h/(h+t)$ (Thyrion, 1969), and Hofmann claims processes. Nevertheless it is simple to adjust any mixed Poisson process with $\lim_{t\to\infty} 1/\theta'(t) = \infty$ to another in which this limit is finite.

Suppose the adjusted $\theta(\cdot)$ is $\theta_+(\cdot)$ such that $\theta_+(t) = \theta(t) - \varepsilon t$ with ε as small as desired, e.g. $\varepsilon = 0.001$,. In fact the new mixed Poisson process is obtained from the original process by adding a pure Poisson process with mean εt. Then

$$\theta'_+(t) = \theta'(t) - \varepsilon, \qquad \theta_+^{(m)}(t) = \theta^{(m)}(t) \qquad m \geq 2$$

and

$$\lim_{t\to\infty} \theta'_+(t) = -\varepsilon$$

so that

$$\bar{p}_1 = -p_1\theta'_+(0)/\varepsilon \qquad \text{and this is finite and positive.}$$

Consider now the expression for $\bar{b}(x, t)$ with adjusted mean

$$\bar{b}(x, t) = \frac{t}{\theta_+(t)}\,\theta'_+(t)\,b(x) + \sum_{m=2}^{\infty} (-1)^m \frac{t^m}{m!}\,\frac{\theta^{(m)}(t)}{-\theta_+(t)}\,b^{m*}(x)$$

Assume that

$$\lim_{t\to\infty} \frac{t^m\theta^{(m)}(t)}{\theta_+(t)} = 0 \qquad m = 2, 3, \ldots \tag{7}$$

This is true, for example, for every negative binomial claim process, where $\theta^{(m)}(t) = (-1)^m(m-1)!h/(h+t)^m$, and for those of Hofmann with $a>1$; this latter inequality is satisfied by one of the two observational results given by Hofmann (1955). Then

$$\lim_{t\to\infty} \bar{b}(x, t) = b(x) \lim_{t\to\infty} \left\{\frac{t}{\theta_+(t)}\,\theta'_+(t)\right\} = b(x) \lim_{t\to\infty} \frac{\theta'_+(t)}{\theta'_+(t)} = b(x)$$

It follows that (1) and (2) for $U(w)$ and $\lim_{t\to\infty}\{1 - U(w)\}$, respectively, can be used *without modification* for any mixed Poisson claim process satisfying (7). It is to be observed that this result, which takes account of the parameter values in $\theta(t)$ only through (7), holds for all positive ε and when $\varepsilon = 0$.

I would like to thank Marc-Henri Amsler for suggesting this research. He pointed out to me that nonlife insurance companies with, perhaps, 100,000 claims in a year were not all that uncommon and that my restriction to small expected claim numbers in Seal (1978) and elsewhere was not the only solution for the practical man.

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Summary

Only mixed (including simple) Poisson claim occurrences and Pareto and lognormal claim size distributions have been justified statistically and this paper on large company ruin is limited to their application. For these cases the asymptotic formulas for the probability of ruin with large initial risk reserve are a failure numerically, and Bartholomew's (1963) simple formula, when applied to Cramér's exact equation (1), produces unacceptable results. Repeated Simpson quadrature of the integral in (1) gives good results at the cost of lengthy calculations. Fitting a Pearson curve to the moments of the survival distribution function is reasonable for the lognormal. Table A summarizes these results. The paper concludes with a proof that negative binomial claims are equivalent to Poisson claims so far as eventual ruin is concerned.

Zusammenfassung

Nur Poisson- oder zusammengesetzte Poissonprozesse für die Schadenzahlen und Pareto- oder lognormale Verteilungen für die Schadenhöhe konnten durch statistische Beobachtungen gerechtfertigt werden. Die vorliegenden Arbeit über die Ruinwahrscheinlichkeit grosser Portefeuilles beschränkt sich auf diese Hypothesen. Für grosse Portefeuilles mit hoher anfänglicher Schwankungsreserve sind die asymptotischen Formeln für die Ruinwahrscheinlichkeit numerisch nicht verwendbar, und Bartholomews einfache Formel (1963), angewandt auf Gleichung (1), liefert unbrauchbare Resultate. Mehrfache Anwendung der Simpsonschen Regel auf das Integral in (1) führt zu guten Ergebnissen, allerdings auf Kosten längerer Berechnungen. Eine Approximation der Momente der Verteilungsfunktion der Überlebenswahrscheinlichkeit durch eine Pearsonkurve ist im lognomalen Fall indessen sinnvoll. Tabelle A fasst die Resultate zusammen. Die Arbeit schliesst mit einem Beweis, dass Schadenzahlen vom Typ negativ binomal und Poisson hinsichtlich der Ruinwahrscheinlichkeit äquivalent sind.

Résumé

Les observations statistiques n'ont justifié l'usage que des processus de type Poisson ou Poisson pondéré pour l'apparition des sinistres et des lois de type Pareto ou lognormal pour le montant des sinistres. Le présent article, consacré à la ruine de gros portefeuilles, se limite à ces hypothèses. Pour de gros portefeuilles avec de fortes provisions de fluctuation, les formules asymptotiques pour la probabilité de ruine ne sont pas praticables numériquement. La formule simple de Bartholomew (1963), appliquée à l'équation exacte de Cramèr (1), produit des résultats inacceptables. Des quadratures répétées selon Simpson sur l'intégrale dans (1) donnent de bons résultats mais entraînent de longs calculs. Pour le cas lognormal, une courbe de Pearson adaptée aux moments de la fonction de distribution de la probabilité de survie conduit à des conclusions raisonnables. Le tableau A résume ces résultats. L'article se termine par une démonstration que les sinistres de type binomial négatif sont équivalents à des sinistres de type Poisson pour ce qui concerne le problème de la ruine.