

# A lower bound for the probability of ruin, given a compound Poisson process

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## A lower bound for the probability of ruin, given a compound Poisson process

For a compound Poisson process  $S_t$ ,  $t \geq 0$ , specified by the Poisson parameter  $\lambda > 0$  and the claim amount distribution  $Q$  on  $(0, \infty)$  the probability of ruin, given the initial reserve  $x \geq 0$  and the total of premiums  $c > 0$  received in the time interval  $(0, 1)$ , is defined by

$$\psi(x) = P\{x + ct - S_t < 0 \text{ for some } t > 0\}.$$

One of the most famous results in ruin theory now is the inequality

$$\psi(x) \leq e^{-Rx}, \quad x \geq 0,$$

where  $R > 0$  is the so-called adjustment coefficient. This result, however, has certain shortages: From the mathematical point of view the underlying assumptions are very strong, i.e. one has to assume that the moment generating function of  $Q$  exists for some  $t_0 > 0$ , and, concerning applications, one has to realize that it is nearly impossible to calculate  $R$  explicitly.

It is the purpose of the present note to give a lower bound for  $\psi(x)$  that can be computed easily. The only assumption which has to be made is a natural one, namely that the mean of  $Q$  exists. The result also appears to be of a certain interest for the user: If the lower bound is too large, then he knows that he has to increase the initial reserve  $x$  in order to obtain a sufficiently small probability of ruin.

**Theorem:** For  $c > \lambda\mu$  and  $x \geq 0$  we have

$$\psi(x) \geq 1 - (1-p) e^{px/\mu} \sum_{i=0}^m \frac{(-1)^i}{i!} \left[ \left( \frac{x}{\mu} - i \right) p e^{-p} \right]^i,$$

where  $\mu$  is the mean of  $Q$ ,

$$m = \text{int} \left( \frac{x}{\mu} \right) \quad \text{and} \quad p = \frac{\lambda\mu}{c}.$$

*Proof:* Since

$$\begin{aligned}\psi(x) &= P\left\{\sup_n \sum_{i=1}^n (X_i - c W_i) > x\right\} \\ &= P\left\{\sup_n \sum_{i=1}^n \left(\frac{X_i}{\mu} - \frac{c}{\mu} W_i\right) > \frac{x}{\mu}\right\}\end{aligned}$$

where  $X_1, X_2, \dots$  are i.i.d. with common distribution  $Q$  and  $W_1, W_2, \dots$  are i.i.d. according to an exponential distribution with mean  $1/\lambda$ , we may assume in the following that

$$\mu = 1$$

(This means that  $X_i, c$  and  $x$  are measured in  $\mu$ -units.) Let  $Q_0$  be the distribution concentrated in  $\mu$ , i.e.  $Q_0\{1\} = 1$ . Then the means of  $Q$  and  $Q_0$  are identical and  $Q$  is more dangerous than  $Q_0$  in the sense of *Bühlmann et al.* [1].

If  $\psi_0(x)$  is the probability of ruin, given the compound Poisson process with Poisson parameter  $\lambda$ , the claim amount distribution  $Q_0$ , and the premium rate  $c > \lambda$ , then by the results in *Bühlmann et al.* [1] and in *Michel* [4]

$$\psi_0(x) \leq \psi(x), \quad x \geq 0.$$

The following proposition therefore gives the assertion.

**Proposition:** We have

$$\psi_0(x) = 1 - (1-p) e^{px} \sum_{i=0}^{[x]} \frac{(-1)^i}{i!} [(x-i)p e^{-p}]^i,$$

where  $p = \lambda/c$  and  $[x] = \text{int}(x)$ .

*Proof:* Here we use the general representation (with  $p = \lambda\mu/c < 1$ )

$$\psi(x) = (1-p) \sum_{k=1}^{\infty} p^k D^{*k}(x, \infty), \quad x \geq 0,$$

where  $D$  is the distribution on  $(0, \infty)$  with the Lebesgue-density

$$y \rightarrow \frac{1}{\mu} Q(y, \infty) 1_{(0, \infty)}(y).$$

(This immediately follows from the renewal equation (3.7) in *Gerber* [3], p. 115, in connection with the known representation of the solution of such an equation.)

In our case, i.e.  $Q_0\{1\} = 1$ ,  $D$  is the uniform distribution on  $(0, 1)$ . Hence, we have for  $k = 1, 2, \dots$ ,

$$D^{*k}[0, x] = \frac{1}{k!} \sum_{i=0}^k (-1)^i \binom{k}{i} [(x-i)^+]^k,$$

where  $a^+ = \max(0, a)$ . This formula can be found in *Feller* [2], Theorem 1, p. 27. Therefore, with  $p = \lambda/c$ ,

$$\begin{aligned} 1 - \psi(x) &= (1-p) \sum_{k=0}^{\infty} p^k D^{*k}[0, x] \\ &= (1-p) \sum_{k=0}^{\infty} p^k \sum_{i=0}^k \frac{(-1)^i}{i!(k-i)!} [(x-i)^+]^k \\ &= (1-p) \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} \sum_{k=i}^{\infty} \frac{p^k}{(k-i)!} [(x-i)^+]^k \\ &= (1-p) \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} [p(x-i)^+]^i e^{p(x-i)^+} \end{aligned}$$

Hence, the result follows.

*Remarks:*

(i) If we write  $\psi_0(x) = \psi_0(x, \lambda, c)$ , then the lower bound in the theorem is  $\psi_0(x/\mu, \lambda, c/\mu)$ , and this in turn is the probability of ruin, given the claim amount distribution which is concentrated in  $\mu$ , the rate of premiums  $c$ , and the initial reserve  $x$ .

Hence, the result of the theorem may be restated as follows: Given  $\lambda, \mu > 0$  and  $c > \lambda\mu$ , the distribution concentrated in  $\mu$  yields the smallest probability of ruin (for all  $x \geq 0$ ) in the class of all claim amount distributions with mean  $\mu$ .

(ii) According to formulas (5.27), p. 124, and (5.29), p. 125, in *Gerber* [3] we have the following asymptotic result for the lower bound in our theorem, say  $b(x)$ ,

$$b(x) \sim C e^{-R_0 x} \quad \text{for } x \rightarrow \infty,$$

where  $R_0$  is the (uniquely determined) positive solution of the equation

$$\lambda e^{r\mu} = \lambda + cr$$

and

$$C = (c - \lambda\mu) / \left( \lambda + cR_0 - \frac{c}{\mu} \right).$$

(Since the denominator represents the derivative of  $r \rightarrow (1/\mu)(\lambda e^{r\mu} - \lambda - cr)$  taken for  $r = R_0$ , it is positive.)

(iii) If the adjustment coefficient  $R$  for  $(Q, \lambda, c)$  exists, i.e. the positive solution of the equation

$$\lambda \int_0^{\infty} e^{rx} Q(dx) = \lambda + cr,$$

and if the mean of  $Q$  is  $\mu$ , then we have

$$R \leq R_0,$$

where  $R_0$  has been defined in (ii), i.e.  $R_0$  is the adjustment coefficient for  $(Q_0, \lambda, c)$ , where  $Q_0$  is the distribution concentrated in  $\mu$ . (This follows from the theorem, (ii), and the fact that  $\psi(x) \leq e^{-Rx}$ ,  $x \geq 0$ .)

Furthermore, the maximum value  $R_0$  for  $R$  (in the class of all distributions with mean  $\mu$ , for which  $R$  exists) is attained only for the distribution  $Q_0$ , i.e. if  $Q$  is different from  $Q_0$ , then we have

$$R < R_0.$$

To see this assume that  $R = R_0$ . Then  $R$  fulfills the equations

$$\lambda \int_0^{\infty} e^{Rx} Q(dx) = \lambda + cR \quad \text{and} \quad \lambda e^{R\mu} = \lambda + cR,$$

i.e. we have

$$\lambda e^{R\mu} = \lambda \int_0^{\infty} e^{Rx} Q(dx).$$

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This, in turn, means equality in *Jensen's* inequality

$$e^{R\mu} \leq \int_0^{\infty} e^{Rx} Q(dx).$$

As  $x \rightarrow e^{Rx}$  is strictly convex, we therefore have

$$x = \int_0^{\infty} y Q(dy) = \mu \quad Q\text{-a.e.},$$

i.e.

$$Q = Q_0.$$

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**Abstract**

Under the assumption that the mean  $\mu$  of the claim size distribution exists we derive a lower bound for the probability of ruin, given a compound Poisson process. This bound cannot be improved as it is attained for the claim size distribution that is concentrated in  $\mu$ . Furthermore, it is shown that the claim size distribution which is concentrated in  $\mu$  admits the largest adjustment coefficient in the class of all claim size distributions with mean  $\mu$  for which the adjustment exists.

**Zusammenfassung**

Hergeleitet wird eine untere Schranke für die Ruinwahrscheinlichkeit, gegeben ein zusammengesetzter Poisson-Prozess, unter der Voraussetzung, dass der Mittelwert  $\mu$  der Schadenhöhenverteilung existiert. Diese Schranke ist insofern nicht verbesserbar, als sie exakt ist für die in  $\mu$  konzentrierte Schadenhöhenverteilung. Ferner wird gezeigt, dass die in  $\mu$  konzentrierte Schadenhöhenverteilung in der Klasse aller Schadenhöhenverteilungen mit Mittelwert  $\mu$ , für die der Anpassungskoeffizient existiert, den grössten Anpassungskoeffizienten besitzt.

**Résumé**

L'article présente une borne inférieure de la probabilité de ruine d'un processus de Poisson sous l'hypothèse de l'existence de la moyenne  $\mu$  du montant d'un sinistre. Cette borne ne peut pas être améliorée en ce sens qu'elle est atteinte dans le cas de la distribution concentrée en  $\mu$ . De plus, l'article montre que, dans la classe des distributions possibles des montants de sinistres de moyenne  $\mu$  et pour lesquelles existe le coefficient d'ajustement, la distribution concentrée en  $\mu$  possède le plus grand coefficient d'ajustement.