Nelson De Pril, Leuven

The Distribution of Actuarial Functions*

1 Introduction

In the classical textbooks on life contingencies only the expected value of actuarial functions has been considered in studying life insurances and annuities. The use of concepts of probability theory such as variance, moment generating function, probability density function and distribution function has been largely avoided. By considering only the mean value of the benefits information is lost on other aspects of the distribution, such as the variability of the actual benefits paid out.

It is the merit of the recent books by *Bowers* et al. (1986) and *Gerber* (1986) that they have stressed the fact that actuarial functions are random variables depending on the time until death (for simplicity the interest rate is taken as constant; this will also be the case in this paper). The much fuller use of probability theory is apparent e.g. in the calculation of the variance of actuarial functions, giving a measure for the actual risk of the insurer.

What we have not found in these books, nor in other publications, is a general and systematic treatment of the distribution of actuarial functions.

Therefore we will give in this paper an explicit expression for the probability density function and the distribution function of the most common actuarial functions on a single life. Our main purpose is not mathematical originality or sophistication, but to give a synthesis of the probabilistic basis of life insurance mathematics.

For completeness both continuous and discrete actuarial functions will be considered. In the first approach the future lifetime is regarded as a continuous random variable, in the second a discrete random variable is associated to it.

2 The distribution of continuous actuarial functions

In the continuous model one considers the continuous non-negative random variable $T(x)$, further abbreviated as T, representing the future lifetime of a life-aged-x.

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Using the common actuarial notation, the distribution function (d.f.) of T can be written as

$$F(t) = Pr[T \leq t] = {}_tq_x = 1 - {}_tp_x \qquad t \geq 0 \tag{1}$$

with

$${}_0q_x = 0 \qquad \text{and} \qquad \lim_{t \to \infty} {}_tq_x = 1\,.$$

The probability density function (p.d.f.) of T is given by

$$f(t) = F'(t) = {}_tp_x \mu_{x+t} \qquad t \geq 0 \tag{2}$$

where μ_x denotes the force of mortality of a life-aged-x.
The present value at policy issue of the benefit payment from a given life insurance contract can be represented by an appropriate function of T. The d.f. and p.d.f. of this non-negative random function S can be found from the distribution of T.
The formulae will be developed for a benefit equal to a unit amount and the following financial quantities will be used:

i :	the constant annual rate of interest, $i > 0$;
$\delta = \ln(1+i)$:	force of interest;
$v = 1/(1+i)$:	present value of 1 due in a year's time;
$\bar{a}_{\overline{n}\rceil} = (1 - v^n)/\delta$:	present value of an annuity-certain of 1 per annum payable continuously for n years.

The characteristics of the distribution of the most common continuous actuarial functions are summarized in the Tables 1 to 3 (see Appendix), respectively dealing with the life insurances, endowment insurances and annuities. In the first column appears the name of the function and the International Actuarial Notation for the corresponding net single premium, that is the expected value of the random variable S which is defined in column 2. The following columns contain the p.d.f. $f(s)$ and the d.f. $F(s)$ of S for values of $s \geq 0$. Since S is a non-negative random variable one has of course $f(s) = F(s) = 0$ for $s < 0$. For some functions one has a mixed distribution with both a probability mass and a continuous part. Proofs are easy and are left to the reader.

3 The distribution of discrete actuarial functions

In this case one associates with future lifetime a discrete random variable $K(x)$, further abbreviated as K, representing the number of full years to death of a person aged x. This random variable is defined over the non-negative integers and one has the relationship $K = [T]$, where the brackets denote the greatest integer function.

The probability function (p.f.) of K can be written as

$$\begin{aligned} f(k) &= Pr[K = k] = Pr[k \leq T < k+1] \\ &= {}_kp_x - {}_{k+1}p_x = {}_kp_x\, q_{x+k} = {}_{k|}q_x \qquad k = 0, 1, \ldots \end{aligned} \tag{3}$$

with ${}_{0|}q_x = q_x$.

The d.f. of K is given by

$$F(k) = Pr[K \leq k] = {}_{[k+1]}q_x \qquad k \geq 0 \tag{4}$$

where $[k+1]$ is the greatest integer less than or equal to $k+1$.

Remark that one has also that

$$Pr[K < k] = {}_{]k[}q_x \qquad k > 0 \tag{5}$$

where $]k[$ denotes the smallest integer greater than or equal to k.

Analogous to the continuous case the discrete actuarial functions are obtained by considering appropriate functions of K. The following symbols from the theory of interest will be used:

$d = 1 - v$:	discount rate;
$\ddot{a}_{\overline{n}\rceil} = (1 - v^n)/d$:	present value of an annuity-due of 1 per annum payable for n years;
$a_{\overline{n}\rceil} = (1 - v^n)/i$:	present value of an immediate annuity of 1 per annum payable for n years, in the limiting case $n = 0$ one has $a_{\overline{0}\rceil} = 0$.

Tables 4 to 7 summarize the results for the distribution of the discrete actuarial functions. They deal respectively with life insurances, endowment insurances, annuities-due and annuities-immediate.

Nelson De Pril
University of Leuven
Institute of Actuarial Science
Dekenstraat 2
B–3000 Leuven
Belgium

References

Bauwelinckx, T. (1988): Verdelingsfuncties voor enkele klassieke levensverzekeringsvormen, unpublished note, University of Leuven.

Bowers, N. L. / Gerber, H. U. / Hickman, J. C. / Jones, D. A. / Nesbitt, C. J. (1986): Actuarial Mathematics. The Society of Actuaries, Itasca, Illinois.

Gerber, H. U. (1986): Lebensversicherungsmathematik. Vereinigung schweizerischer Versicherungsmathematiker, Springer-Verlag, Berlin / Heidelberg.

Appendix

Table 1

Life insurances payable at the moment of death

$$\tau = -\frac{1}{\delta}\ln s$$

Life insurance	S		$f(s)$		$F(s)$	
whole life			${}_{\tau}p_x\mu_{x+\tau}/\delta s$	$0 < s \le 1$	0	$s = 0$
	v^T	$T \ge 0$			${}_{\tau}p_x$	$0 < s < 1$
$\bar{A}_x$			0	elsewhere	1	$s \ge 1$
n-year term	v^T	$0 \le T < n$	${}_np_x$	$s = 0$	${}_np_x$	$0 \le s \le v^n$
			${}_{\tau}p_x\mu_{x+\tau}/\delta s$	$v^n < s \le 1$	${}_{\tau}p_x$	$v^n < s < 1$
$\bar{A}^{\ 1}_{x:\overline{n\rvert}}$	0	$T \ge n$	0	elsewhere	1	$s \ge 1$
m-year deferred	0	$0 \le T < m$	${}_mq_x$	$s = 0$	${}_mq_x$	$s = 0$
			${}_{\tau}p_x\mu_{x+\tau}/\delta s$	$0 < s \le v^m$	${}_mq_x + {}_{\tau}p_x$	$0 < s < v^m$
${}_{m\rvert}\bar{A}_x$	v^T	$T \ge m$	0	elsewhere	1	$s \ge v^m$
m-year deferred	0	$0 \le T < m$	${}_mq_x + {}_{m+n}p_x$	$s = 0$	${}_mq_x + {}_{m+n}p_x$	$0 \le s \le v^{m+n}$
n-year term	v^T	$m \le T < m+n$	${}_{\tau}p_x\mu_{x+\tau}/\delta s$	$v^{m+n} < s \le v^m$	${}_mq_x + {}_{\tau}p_x$	$v^{m+n} < s < v^m$
${}_{m\rvert}\bar{A}^{\ 1}_{x:\overline{n\rvert}}$	0	$T \ge m+n$	0	elsewhere	1	$s \ge v^m$

Table 2

Endowment insurances with death benefits payable at the moment of death

$$\tau = -\frac{1}{\delta} \ln s$$

Endowment insurance	S		$f(s)$		$F(s)$	
n-year pure	0	$0 \leq T < n$	${}_nq_x$	$s = 0$	${}_nq_x$	$0 \leq s < v^n$
endowment			${}_np_x$	$s = v^n$		
$\bar{A}_{x:\overline{n}\rvert}^{\;\;1}$ or ${}_nE_x$	v^n	$T \geq n$	0	elsewhere	1	$s \geq v^n$
n-year endowment	v^T	$0 \leq T < n$	${}_np_x$	$s = v^n$	0	$0 \leq s < v^n$
			${}_\tau p_x \mu_{x+\tau}/\delta s$	$v^n < s \leq 1$	${}_\tau p_x$	$v^n \leq s < 1$
$\bar{A}_{x:\overline{n}\rvert}$	v^n	$T \geq n$	0	elsewhere	1	$s \geq 1$
m-year deferred	0	$0 \leq T < m$	${}_mq_x$	$s = 0$	${}_mq_x$	$0 \leq s < v^{m+n}$
n-year endowment	v^T	$m \leq T < m+n$	${}_{m+n}p_x$	$s = v^{m+n}$	${}_mq_x + {}_\tau p_x$	$v^{m+n} \leq s < v^m$
${}_{m\rvert}\bar{A}_{x:\overline{n}\rvert}$	v^{m+n}	$T \geq m+n$	${}_\tau p_x \mu_{x+\tau}/\delta s$	$v^{m+n} < s \leq v^m$	1	$s \geq v^m$
			0	elsewhere		

Table 3

Continuous life annuities

$$\alpha = -\frac{1}{\delta}\ln(1-\delta s) \qquad \text{and} \qquad \beta = -\frac{1}{\delta}\ln(v^m - \delta s)$$

Life annuity	S		$f(s)$		$F(s)$	
whole life			${}_{\alpha}p_x \mu_{x+\alpha}/(1-\delta s)$	$0 \le s < 1/\delta$	${}_{\alpha}q_x$	$0 \le s < 1/\delta$
	$\bar{a}_{\overline{T}\rceil}$	$T \ge 0$				
$\bar{a}_x$			0	elsewhere	1	$s \ge 1/\delta$
n-year temporary	$\bar{a}_{\overline{T}\rceil}$	$0 \le T < n$	${}_{\alpha}p_x \mu_{x+\alpha}/(1-\delta s)$	$0 \le s < \bar{a}_{\overline{n}\rceil}$	${}_{\alpha}q_x$	$0 \le s < \bar{a}_{\overline{n}\rceil}$
			${}_{n}p_x$	$s = \bar{a}_{\overline{n}\rceil}$		
$\bar{a}_{x:\overline{n}\rceil}$	$\bar{a}_{\overline{n}\rceil}$	$T \ge n$	0	elsewhere	1	$s \ge \bar{a}_{\overline{n}\rceil}$
m-year deferred	0	$0 \le T < m$	${}_{m}q_x$	$s = 0$	${}_{\beta}q_x$	$0 \le s < v^m/\delta$
			${}_{\beta}p_x \mu_{x+\beta}/(v^m-\delta s)$	$0 < s < v^m/\delta$		
${}_{m\mid}\bar{a}_x$	$\bar{a}_{\overline{T}\rceil} - \bar{a}_{\overline{m}\rceil}$	$T \ge m$	0	elsewhere	1	$s \ge v^m/\delta$
m-year deferred	0	$0 \le T < m$	${}_{m}q_x$	$s = 0$	${}_{\beta}q_x$	$0 \le s < v^m\bar{a}_{\overline{n}\rceil}$
n-year temporary	$\bar{a}_{\overline{T}\rceil} - \bar{a}_{\overline{m}\rceil}$	$m \le T < m+n$	${}_{\beta}p_x \mu_{x+\beta}/(v^m-\delta s)$	$0 < s < v^m\bar{a}_{\overline{n}\rceil}$		
${}_{m\mid}\bar{a}_{x:\overline{n}\rceil}$	$v^m\bar{a}_{\overline{n}\rceil}$	$T \ge m+n$	${}_{m+n}p_x$	$s = v^m\bar{a}_{\overline{n}\rceil}$	1	$s \ge v^m\bar{a}_{\overline{n}\rceil}$
			0	elsewhere		

Table 4

Life insurance payable at the end of the year of death

$$\tau = \left] -1 - \frac{1}{\delta} \ln s \right[$$

Life insurance	S	$f(s)$	$F(s)$
whole life A_x	v^{K+1} $\quad K = 0, 1, \ldots$	${}_{k\|}q_x$ $\quad s = v^{k+1}$, $k = 0, 1, \ldots$ 0 $\quad$ elsewhere	0 $\quad s = 0$ ${}_{\tau}p_x$ $\quad 0 < s < v$ 1 $\quad s \geq v$
n-year term $A^{\,1}_{x:\overline{n\|}}$	v^{K+1} $\quad K = 0, 1, \ldots, n-1$ 0 $\quad K = n, n+1, \ldots$	${}_{n}p_x$ $\quad s = 0$ ${}_{k\|}q_x$ $\quad s = v^{k+1}$, $k = 0, 1, \ldots, n-1$ 0 $\quad$ elsewhere	${}_{n}p_x$ $\quad 0 \leq s < v^n$ ${}_{\tau}p_x$ $\quad v^n \leq s < v$ 1 $\quad s \geq v$
m-year deferred ${}_{m\|}A_x$	0 $\quad K = 0, 1, \ldots, m-1$ v^{K+1} $\quad K = m, m+1, \ldots$	${}_{m}q_x$ $\quad s = 0$ ${}_{k\|}q_x$ $\quad s = v^{k+1}$, $k = m, m+1, \ldots$ 0 $\quad$ elsewhere	${}_{m}q_x$ $\quad s = 0$ ${}_{m}q_x + {}_{\tau}p_x$ $\quad 0 < s < v^{m+1}$ 1 $\quad s \geq v^{m+1}$
m-year deferred n-year term ${}_{m\|}A^{\,1}_{x:\overline{n\|}}$	0 $\quad K = 0, 1, \ldots m-1$ v^{K+1} $\quad K = m, \ldots, m+n-1$ 0 $\quad K = m+n, m+n+1, \ldots$	${}_{m}q_x + {}_{m+n}p_x$ $\quad s = 0$ ${}_{k\|}q_x$ $\quad s = v^{k+1}$, $k = m, \ldots, m+n-1$ 0 $\quad$ elsewhere	${}_{m}q_x + {}_{m+n}p_x$ $\quad 0 \leq s < v^{m+n}$ ${}_{m}q_x + {}_{\tau}p_x$ $\quad v^{m+n} \leq s < v^{m+1}$ 1 $\quad s \geq v^{m+1}$

Table 5

Endowment insurances with death benefits payable at the end of the year of death

$$\tau = \left]-1-\frac{1}{\delta}\ln s\right[$$

Endowment insurance	S		$f(s)$		$F(s)$	
n-year pure endowment	0	$K = 0, 1, \ldots, n-1$	${}_nq_x$	$s = 0$	${}_nq_x$	$0 \leq s < v^n$
			${}_np_x$	$s = v^n$		
$A_{x:\overline{n\rceil}}^{\ \ 1}$ or ${}_nE_x$	v^n	$K = n, n+1, \ldots$	0	elsewhere	1	$s \geq v^n$
n-year endowment	v^{K+1}	$K = 0, 1, \ldots, n-1$	${}_{n-1}p_x$	$s = v^n$	0	$0 \leq s < v^n$
			${}_{k\mid}q_x$	$s = v^{k+1}$, $k = 0, 1, \ldots, n-2$	${}_\tau p_x$	$v^n \leq s < v$
$A_{x:\overline{n\rceil}}$	v^n	$K = n, n+1, \ldots$	0	elsewhere	1	$s \geq v$
m-year deferred	0	$K = 0, 1, \ldots, m-1$	${}_mq_x$	$s = 0$	${}_mq_x$	$0 \leq s < v^{m+n}$
n-year endowment	v^{K+1}	$K = m, \ldots, m+n-1$	${}_{m+n-1}p_x$	$s = v^{m+n}$	${}_mp_x + {}_\tau p_x$	$v^{m+n} \leq s < v^{m+1}$
${}_{m\mid}A_{x:\overline{n\rceil}}$	v^{m+n}	$K = m+n, m+n+1, \ldots$	${}_{k\mid}q_x$	$s = v^{k+1}$, $k = m, \ldots, m+n-2$	1	$s \geq v^{m+1}$
			0	elsewhere		

Table 6

Life annuities-due

$$\alpha = \left[-\frac{1}{\delta}\ln(1-ds)\right] \qquad \text{and} \qquad \beta = \left[-\frac{1}{\delta}\ln(v^m-ds)\right]$$

Life annuity	S		$f(s)$		$F(s)$	
whole life			${}_{k\mid}q_x$	$s=\ddot{a}_{\overline{k+1}\rceil}$, $k=0,1,\dots$	${}_{\alpha}q_x$	$0\le s<1/d$
	$\ddot{a}_{\overline{K+1}\rceil}$	$K=0,1,\dots$				
$\ddot{a}_x$			0	elsewhere	1	$s\ge 1/d$
n-year temporary	$\ddot{a}_{\overline{K+1}\rceil}$	$K=0,1,\dots,n-1$	${}_{k\mid}q_x$	$s=\ddot{a}_{\overline{k+1}\rceil}$, $k=0,1,\dots,n-2$	${}_{\alpha}q_x$	$0\le s<\ddot{a}_{\overline{n}\rceil}$
			${}_{n-1}p_x$	$s=\ddot{a}_{\overline{n}\rceil}$		
$\ddot{a}_{x:\overline{n}\rceil}$	$\ddot{a}_{\overline{n}\rceil}$	$K=n,n+1,\dots$	0	elsewhere	1	$s\ge\ddot{a}_{\overline{n}\rceil}$
m-year deferred	0	$K=0,1,\dots,m-1$	${}_{m}q_x$	$s=0$	${}_{\beta}q_x$	$0\le s<v^m/d$
			${}_{m+k\mid}q_x$	$s=v^m\ddot{a}_{\overline{k+1}\rceil}$, $k=0,1,\dots$		
${}_{m\mid}\ddot{a}_x$	$\ddot{a}_{\overline{K+1}\rceil}-\ddot{a}_{\overline{m}\rceil}$	$K=m,m+1,\dots$	0	elsewhere	1	$s\ge v^m/d$
m-year deferred	0	$K=0,1,\dots,m-1$	${}_{m}q_x$	$s=0$	${}_{\beta}q_x$	$0\le s<v^m\ddot{a}_{\overline{n}\rceil}$
n-year temporary	$\ddot{a}_{\overline{K+1}\rceil}-\ddot{a}_{\overline{m}\rceil}$	$K=m,\dots,m+n-1$	${}_{m+k\mid}q_x$	$s=v^m\ddot{a}_{\overline{k+1}\rceil}$, $k=0,1,\dots,n-2$		
${}_{m\mid}\ddot{a}_{x:\overline{n}\rceil}$	$v^m\ddot{a}_{\overline{n}\rceil}$	$K=m+n,m+n+1,\dots$	${}_{m+n-1}p_x$	$s=v^m\ddot{a}_{\overline{n}\rceil}$	1	$s\ge v^m\ddot{a}_{\overline{n}\rceil}$
			0	elsewhere		

Table 7

Life annuities-immediate

$$\alpha = \left[-\frac{1}{\delta} \ln(v - ds) \right] \quad \text{and} \quad \beta = \left[-\frac{1}{\delta} \ln(v^{m+1} - ds) \right]$$

Life annuity	S	$f(s)$	$F(s)$
whole life a_x	$a_{\overline{K}\rceil}$ $\quad K = 0, 1, \ldots$	${}_{k\vert}q_x$ $\quad s = a_{\overline{k}\rceil}$, $k = 0, 1, \ldots$ 0 $\quad$ elsewhere	${}_{\alpha}q_x$ $\quad 0 \le s < 1/i$ 1 $\quad s \ge 1/i$
n-year temporary $a_{x:\overline{n}\rceil}$	$a_{\overline{K}\rceil}$ $\quad K = 0, 1, \ldots, n-1$ $a_{\overline{n}\rceil}$ $\quad K = n, n+1, \ldots$	${}_{k\vert}q_x$ $\quad s = a_{\overline{k}\rceil}$, $k = 0, 1, \ldots, n-1$ ${}_{n}p_x$ $\quad s = a_{\overline{n}\rceil}$ 0 $\quad$ elsewhere	${}_{\alpha}q_x$ $\quad 0 \le s < a_{\overline{n}\rceil}$ 1 $\quad s \ge a_{\overline{n}\rceil}$
m-year deferred ${}_{m\vert}a_x$	0 $\quad K = 0, 1, \ldots, m-1$ $a_{\overline{K}\rceil} - a_{\overline{m}\rceil}$ $\quad K = m, m+1, \ldots$	${}_{m+1}q_x$ $\quad s = 0$ ${}_{m+k\vert}q_x$ $\quad s = v^m a_{\overline{k}\rceil}$, $k = 1, 2, \ldots$ 0 $\quad$ elsewhere	${}_{\beta}q_x$ $\quad 0 \le s < v^m/i$ 1 $\quad s \ge v^m/i$
m-year deferred n-year temporary ${}_{m\vert}a_{x:\overline{n}\rceil}$	0 $\quad K = 0, 1, \ldots, m-1$ $a_{\overline{K}\rceil} - a_{\overline{m}\rceil}$ $\quad K = m, \ldots, m+n-1$ $v^m a_{\overline{n}\rceil}$ $\quad K = m+n, m+n+1, \ldots$	${}_{m+1}q_x$ $\quad s = 0$ ${}_{m+k\vert}q_x$ $\quad s = v^m a_{\overline{k}\rceil}$, $k = 1, 2, \ldots, n-1$ ${}_{m+n}p_x$ $\quad s = v^m a_{\overline{n}\rceil}$ 0 $\quad$ elsewhere	${}_{\beta}q_x$ $\quad 0 \le s < v^m a_{\overline{n}\rceil}$ 1 $\quad s \ge v^m a_{\overline{n}\rceil}$