From the convolution of uniform distributions to the probability of ruin

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1. Introduction

Let $y_1, y_2, \dots, y_n$ be positive numbers. We consider the uniform distribution over $(0, y_i)$; the corresponding cumulative distribution function is

$$F_i(x) = \begin{cases} 0 & for\ x < 0 \\ \dfrac{x}{y_i} & for\ 0 \leq x < y_i \\ 1 & for\ x \geq y_i \end{cases} \tag{1}$$

$i = 1, 2, \dots, n$. The cumulative distribution function of their convolution is

$$H_n(x) = F_1 * F_2 * \cdots * F_n(x). \tag{2}$$

Explicit formulas for $H_n(x) = H_n(x; y_1, \dots, y_n)$ and its derivative are well known and have been derived in different ways, see for example *Seal* (1950) and *Shiu* (1987). In this note, which is mostly of a pedagogical nature, we shall first show by geometric reasoning how two dual expressions for H_n can be obtained. In section 4 the extension to the convolution of distributions with decreasing probability density functions is discussed. Thanks to these more general formulas certain series expressions for the probability of ruin can be derived easily.

2. Geometric reasoning: the rocket principle

We start with $n = 2$ and prefer to consider $y_1 y_2 H_2(x)$ (instead of $H_2(x)$), since this expression has a geometric interpretation: It is the area of that part of the rectangle

$$\{(x_1, x_2) \mid 0 \leq x_1 \leq y_1, \quad 0 \leq x_2 \leq y_2\}$$

Figure 1a Geometric derivation of formula (3 a)

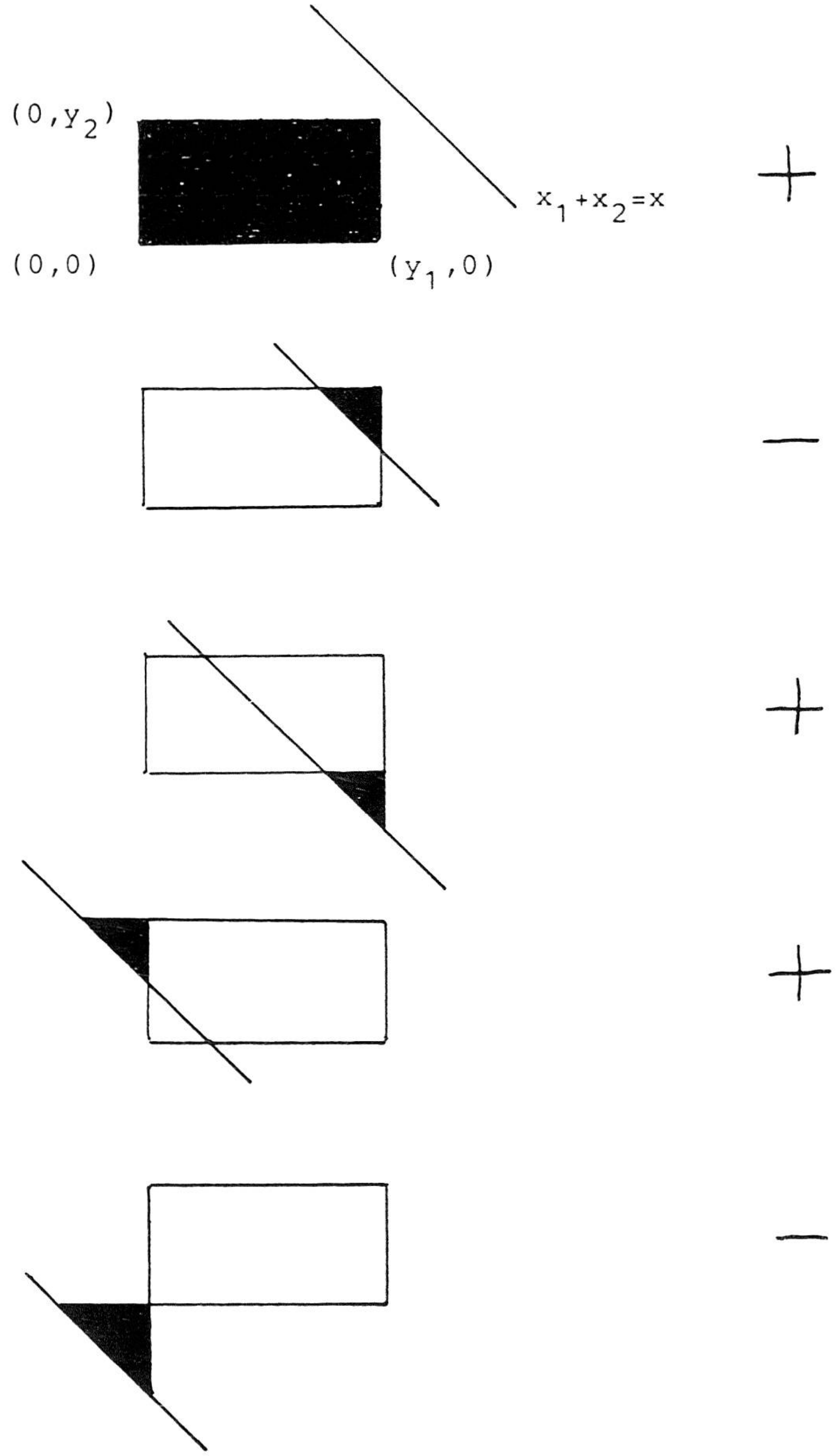
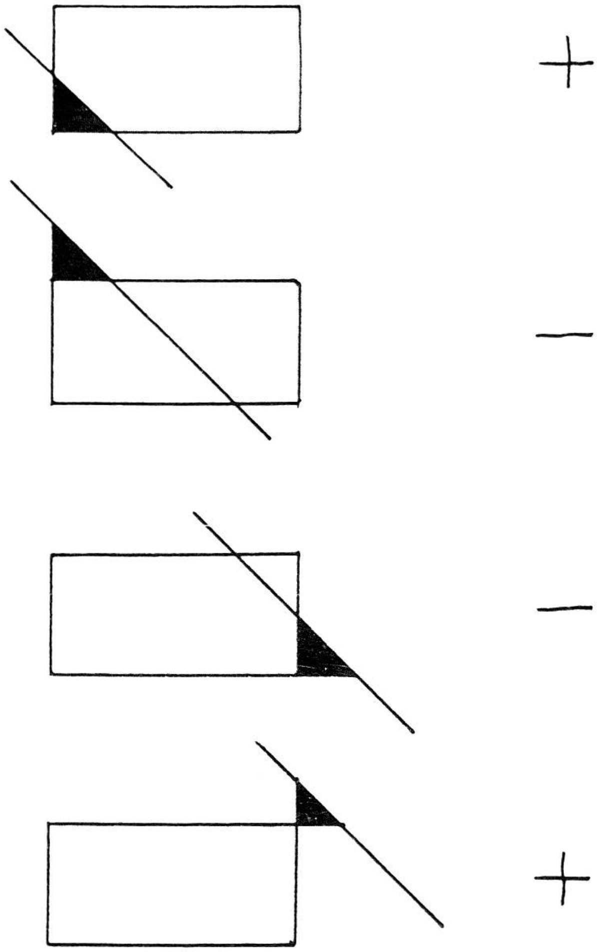


Figure 1b Geometric derivation of formula (3 b)



that is below the line $x_1 + x_2 = x$. There are two formulas for $y_1 y_2 H_2(x)$:

$$\begin{aligned} y_1 y_2 H_2(x) = y_1 y_2 &- \frac{1}{2}(y_1 + y_2 - x)_+^2 \\ &+ \frac{1}{2}(y_1 - x)_+^2 + \frac{1}{2}(y_2 - x)_+^2 \\ &- \frac{1}{2}(-x)_+^2, \end{aligned} \tag{3a}$$

and

$$y_1 y_2 H_2(x) = \frac{1}{2}(x)_+^2 - \frac{1}{2}(x - y_1)_+^2 - \frac{1}{2}(x - y_2)_+^2 + \frac{1}{2}(x - y_1 - y_2)_+^2 . \tag{3b}$$

These inclusion-exclusion type formulas are best derived geometrically. The geometric derivation of (3a), term by term, is illustrated in Figure 1a. The idea is to start with a line above the rectangle (i.e., $x > y_1 + y_2$), to lower the line successively and thereby to make the necessary corrections by adding and subtracting the areas of certain isosceles right triangles. In Figure 1b we illustrate the geometric derivation of (3b). Here, the idea is to start with a line that is below the rectangle and to raise it successively.

For $n = 3$ we consider $y_1 y_2 y_3 H_3(x)$, which is the volume of that part of the three dimensional rectangle

$$\{(x_1, x_2, x_3) \mid 0 \leq x_i \leq y_i, \quad i = 1, 2, 3\}$$

that is below the plane $x_1 + x_2 + x_2 = x$. If we start with a plane that is above the three-dimensional rectangle ($x > y_1 + y_2 + y_3$) and lower the plane successively, we have to add and subtract the volumes of certain isosceles right pyramids. This way we get

$$\begin{aligned} y_1 y_2 y_3 H_3(x) = & y_1 y_2 y_3 \\ & - \frac{1}{6}(y_1 + y_2 + y_3 - x)_+^3 \\ & + \frac{1}{6}(y_1 + y_2 - x)_+^3 + \frac{1}{6}(y_1 + y_3 - x)_+^3 + \frac{1}{6}(y_2 + y_3 - x)_+^3 \\ & - \frac{1}{6}(y_1 - x)_+^3 - \frac{1}{6}(y_2 - x)_+^3 - \frac{1}{6}(y_3 - x)_+^3 \\ & + \frac{1}{6}(-x)_+^3. \end{aligned} \tag{4a}$$

If we start with a plane that is below the three-dimensional rectangle ($x < 0$) and raise it successively, we get

$$\begin{aligned} y_1 y_2 y_3 H_3(x) = & \frac{1}{6}(x)_+^3 \\ & - \frac{1}{6}(x - y_3)_+^3 - \frac{1}{6}(x - y_2)_+^3 - \frac{1}{6}(x - y_1)_+^3 \\ & + \frac{1}{6}(x - y_2 - y_3)_+^3 + \frac{1}{6}(x - y_1 - y_3)_+^3 + \frac{1}{6}(x - y_1 - y_2)_+^3 \\ & - \frac{1}{6}(x - y_1 - y_2 - y_3)_+^3 . \end{aligned} \tag{4b}$$

The artistic reader is invited to draw diagrams similar to those of Figure 1!

3. The general formulas

For arbitrary n, the formulas are now the following natural extensions:

$$\begin{aligned}\left(\prod_{i=1}^{n} y_i\right) H_n(x) = &\prod_{i=1}^{n} y_i \\ &- \frac{1}{n!}(y_1 + \cdots + y_n - x)_+^n \\ &+ \frac{1}{n!}\sum (y_{i_1} + \cdots + y_{i_{n-1}} - x)_+^n \\ &- \frac{1}{n!}\sum (y_{i_1} + \cdots + y_{i_{n-2}} - x)_+^n \\ &\cdots \\ &\pm \frac{1}{n!}(-x)_+^n\end{aligned} \tag{5a}$$

and

$$\begin{aligned}\left(\prod_{i=!}^{n} y_i\right) H_n(x) = &\frac{1}{n!}(x)_+^n \\ &- \frac{1}{n!}\sum_{i=1}^{n}(x - y_i)_+^n \\ &+ \frac{1}{n!}\sum (x - y_{i_1} - y_{i_2})_+^n \\ &\cdots \\ &\pm \frac{1}{n!}(x - y_1 - \cdots - y_n)_+^n .\end{aligned} \tag{5b}$$

These formulas can be verified by induction with respect to n. For (5b) this can be done, for example, by using the recursive relation

$$\left(\prod_{i=1}^{n} y_i\right) H_n(x) = \prod_{i=1}^{n-1} y_i \int_0^{y_n} H_{n-1}(x - y)\, dy \tag{6}$$

and observing that

$$\frac{1}{(n-1)!}\int_0^{y_n} (x - a - y_{n-1})_+^{n-1}\, dy = \frac{1}{n!}(x - a - y_n)_+^n - \frac{1}{n!}(x - a)_+^n \tag{7}$$

for any $a \geq 0$.

4. Application: The convolution of distributions with decreasing probability density functions

Let $Y_1, Y_2, \ldots, Y_n$ be independent, positive random variables. We denote the cumulative distribution function of Y_i by $P_i(x)$ and suppose that its mean, μ_i is finite. To $P_i(x)$ we associate another cumulative distribution function, given by the formula

$$Q_i(x) = \frac{1}{\mu_i} \int_0^x [1 - P_i(z)]\, dz. \tag{8}$$

Note that there is a one-to-one correspondence between Q_i and P_i : as $Q_i(x)$ is a distribution function with decreasing probability density function, the underlying P_i can be retrieved by the formula

$$P_i(x) = 1 - \frac{Q_i'(x)}{Q_i'(0)}. \tag{9}$$

Furthermore, the Laplace transform of Q_i can be expressed in terms of the Laplace transform of P_i:

$$\int_0^\infty e^{-tx} Q_i(dx) = \frac{1 - \int_0^\infty e^{-tx} P_i(dx)}{t\mu_i}. \tag{10}$$

The proof of this well known formula is a simple exercise of integration by parts.

Let us now consider random variables $X_1, \ldots, X_n$ which (given $Y_1, \ldots, Y_n$) are conditionally independent, such that X_i is uniformly distributed over $(0, Y_i)$; then $H_n(x; Y_1, \ldots, Y_n)$ is the conditional cumulative distribution function of $X_1 + \cdots + X_n$. The following Lemma will allow us to apply the results of section 3.

Lemma:

$$E\left[\left(\prod_{i=1}^n Y_i\right) H_n(x; Y_1, \ldots Y_n)\right] = \left(\prod_{i=1}^n \mu_i\right) Q_1 * Q_2 * \cdots * Q_n(x).$$

Proof: We verify this formula by calculating the Laplace transforms. Since

$$\int_0^\infty e^{-tx} H_n(dx; y_1, \ldots y_n) = \prod_{i=1}^n \left(\frac{1 - e^{-ty_i}}{ty_i} \right), \tag{11}$$

we get

$$\begin{aligned}
&\int_0^\infty e^{-tx} E\left[\left(\prod_{i=1}^n Y_i \right) H_n(dx; Y_1, \ldots, Y_n) \right] \\
&\quad = E\left[\int_0^\infty e^{-tx} \left(\prod_{i=1}^n Y_i \right) H_n(dx; Y_1, \ldots Y_n) \right] \\
&\quad = E\left[\prod_{i=1}^n \left(\frac{1 - e^{-tY_i}}{t} \right) \right] \\
&\quad = \prod_{i=1}^n \left(\frac{1 - E\left[e^{-tY_i}\right]}{t} \right) \\
&\quad = \left(\prod_{i=1}^n \mu_i \right) \prod_{i=1}^n \left(\frac{1 - E\left[e^{-tY_i}\right]}{\mu_i t} \right).
\end{aligned} \tag{12}$$

Q.E.D.

We consider now the special case where the Y_i's are identically distributed. We write P, μ, and Q instead of P_i, μ_i, and Q_i, and set

$$S_k = Y_1 + \cdots + Y_k \tag{13}$$

for $k = 1, \ldots, n$ $(S_0 = 0)$. Let us look at the expression

$$\begin{aligned}
&\left(\prod_{i=1}^n Y_i \right) Pr(X_1 + \cdots + X_n \leq x \mid Y_1, \ldots, Y_n) \\
&\quad = \left(\prod_{i=1}^n Y_i \right) H_n(x; Y_1, \ldots, Y_n).
\end{aligned} \tag{14}$$

Now we substitute (5a) or (5b), with y_i replaced by Y_i. Finally we take expectations and use the Lemma to obtain

$$\mu^n Q^{*n}(x) = \mu^n - \frac{1}{n!} \sum_{j=0}^n (-1)^j \binom{n}{j} E\left[(S_{n-j} - x)_+^n \right] \tag{15a}$$

and

$$\mu^n Q^{*n}(x) = \frac{1}{n!} \sum_{j=0}^{n} (-1)^j \binom{n}{j} E\left[(x - S_j)_+^n\right]. \tag{15b}$$

Formula (15a) can also be found in *Gerber* (1988) where it has been derived by probabilistic methods.

5. Application: The probability of ruin

We consider the classical model of the collective theory of risk, where

$$U(t) = u + ct - S(t), \qquad t \geq 0, \tag{16}$$

is the surplus of an insurance company at time t. Here u is the initial surplus, c the rate at which the premiums are received, and

$$S(t) = Y_1 + \cdots + Y_{N(t)} \tag{17}$$

the aggregate claims, a compound Poisson process; the claim amounts $Y_1, Y_2, \ldots$ are i.i.d. random variables with common distribution function $P(x)$, mean μ, and $\{N(t)\}$ is a Poisson process with parameter λ. It is assumed $c > \lambda\mu$.

The probability of ruin, $\psi(u)$, can be expressed by the convolution formula,

$$\psi(u) = (1 - a\mu) \sum_{n=1}^{\infty} (a\mu)^n [1 - Q^{*n}(u)], \tag{18}$$

where $a = \lambda/c$ and

$$Q(x) = \frac{1}{\mu} \int_0^x [1 - P(t)]\, dt \tag{19}$$

in agreement with the notation of section 4.

Thanks to formulas (15) we can now derive two alternative formulas for the probability of ruin. First we replace $1 - Q^{*n}(u)$ in (18) by using (15a). After simplification of the resulting double summation we get

$$\psi(u) = (1 - a\mu) \sum_{k=1}^{\infty} \frac{a^k}{k!} E\left[(S_k - u)_+^k e^{-a(S_k - u)}\right]. \tag{20a}$$

Alternatively we may substitute (15b) into (18). After simplification we obtain the formula

$$1 - \psi(u) = (1 - a\mu) \sum_{k=0}^{\infty} \frac{(-a)^k}{k!} E\left[(u - S_k)_+^k e^{a(u-S_k)}\right]. \tag{20b}$$

In the special case where all claims are of size μ, $S_k = k\mu$, this formula is well known, sometimes by the name of Khintchine-Pollaczek, see *Feller* (1966, formula (2.11) of XIV.3). The general expression (20 b) has been derived by *Shiu* (1988). Formula (20 a) has been derived by *Prabhu* (1965, formula (5.55)). If we add (20 a) and (20 b) we obtain the identity

$$1 = (1 - a\mu) \sum_{k=0}^{\infty} \frac{a^k}{k!} E\left[(S_k - u)^k e^{-a(S_k-u)}\right], \tag{21}$$

which is the starting point of *Gerber* (1988).

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Summary

The dual formulas for the convolution of uniform distributions can be obtained by a geometric reasoning of the inclusion-exclusion type. The two formulas can be generalized to the convolution of distributions with decreasing probability density functions. Finally, this result can be used to get two series expressions for the probability of ruin.

Zusammenfassung

Zwei klassische Formeln für die Faltung von Gleichverteilungen können leicht auf geometrische Weise (gemäss dem "Raketenprinzip", d.h. mit sukkzessiven Korrekturen) hergeleitet werden. Die beiden Formeln werden verallgemeinert für den Fall der Faltung von Verteilungen mit abnehmender Dichtefunktion. Dieses Resultat kann man wiederum benützen, um zwei duale Reihenentwicklungen der Ruinwahrscheinlichkeit herzuleiten.

Résumé

On peut obtenir des formules duales pour la convolution de distributions uniformes en utilisant une approche géometrique de type inclusion-exclusion. On généralise les deux formules au cas de convolution de distributions dont la fonction de densité de probabilité est une fonction décroissante. Finalement, on utilise ce dernier résultat pour développer des expressions duales (sous forme de séries) pour la probabilité de ruine.