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Autor:	Marazzi, Alfio						
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# ALFIO MARAZZI, Lausanne

# Restricted Minimax Credibility: Two Special Cases

## 1 Introduction

The simplest credibility formula  $\delta(y) = ay + b$ , where y is the average claim amount or loss ratio for a contract with risk parameter  $\theta$ , can be derived within a decision-theoretical framework. Indeed, using a quadratic loss function,  $\delta(y)$ is the linear Bayes estimate of  $E(y \mid \theta)$  (see *Bühlmann* [3]). Moreover, it is the exact Bayes estimate of  $E(y \mid \theta)$  when the density  $f(\cdot \mid \theta)$  of y belongs to the single parameter exponential family and the structure function  $U(\theta)$  is the corresponding conjugate prior distribution (see *Jewell* [10]).

Within this framework, two sources of error can distort the performances of the credibility estimate: (a) an inappropriate structure function; (b) an unexpected high frequency of very large claims i.e. an inappropriate model  $f(\cdot | \theta)$ . Minimax credibility was suggested by *Bühlmann* [4] and *Marazzi* [11] as a remedy for (a) and data trimming has been used by *Gisler* [6] in order to deal with (b).

We are going to consider the very simple model  $y = \theta + e$  as an example and will show how the restricted Bayes and minimax principles proposed by *Hodges/Lehmann* [7] can be applied in order to obtain robust estimates of  $\theta$  when: (a) the "error" *e* follows a Gaussian distribution and the structure function is not exactly known; (b) the structure function is Gaussian and the specified error distribution is not accurate. The solutions of the corresponding optimality problems provide a decision-theoretical justification for the wellknown data trimming procedures.

The method has been described for the linear model in *Marazzi* [12]. This paper focuses on two simple special cases and indicates possible extensions where the Gaussian distribution is replaced by the exponential family.

# 1.1 The restricted Bayes and minimax principles

In a decision problem let the unknown parameter  $\theta$  be a random variable with prior distribution U (structure function). Let  $R(\theta, \delta)$  denote the risk

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function of a decision procedure  $\delta$ , for example an estimator of  $E(y \mid \theta)$ , and let  $r(U, \delta) = \int R(\theta, \delta) dU(\theta)$  be the mean Bayes risk.

The Hodges & Lehmann approach to the problem of optimal decisions utilizes the available prior information but, at the same time, provides a safeguard in case this information is not correct. It is motivated as follows: the minimax decision does not use the prior information at all and is associated with the smallest possible value m for the maximum of the risk function; but we may be willing to tolerate a somewhat bigger maximum  $m + c_0 > m$  if, in case the guess at  $\theta$  has been a good one, this results in a substantial decrease in the average risk.

This leads to the following problems:

*P1: The restricted Bayes problem.* Let  $c_0 > 0$  be a given number and  $U_0$  a given prior distribution. Minimize  $r(U_0, \delta)$  subject to

$$R(\theta, \delta) \le m + c_0$$
, for all  $\theta$ .

*PII*: The restricted minimax problem. Let  $\varepsilon \in (0, 1)$  be a given number,  $U_0$  a given prior distribution, and let

$$\mathscr{P}_{\varepsilon} = \{ U \mid U = (1 - \varepsilon)U_0 + \varepsilon H, \quad H \in \mathscr{H} \}.$$

Find  $\delta_{\varepsilon}$  such that  $\sup_{\mathscr{P}_{\varepsilon}} r(U, \delta_{\varepsilon}) = \inf_{\mathscr{D}} \sup_{\mathscr{P}_{\varepsilon}} r(U, \delta)$ .

Here  $\mathscr{H}$  is the set of all prior distributions and  $\mathscr{D}$  is a given class of decision functions. The elements of  $\mathscr{H}$  are sometimes called *contaminations*.

Under general conditions  $\delta_{\varepsilon}$  is Bayes for a least favorable (l.f.) distribution  $U_{\varepsilon}$  in  $\mathscr{P}_{\varepsilon}$  and  $(U_{\varepsilon}, \delta_{\varepsilon})$  is a saddlepoint of the game  $(\mathscr{P}_{\varepsilon}, \mathscr{D}, r)$ . Furthermore, the two restricted problems are equivalent in the following sense: if  $\delta_{\varepsilon}$  is restricted minimax, then  $\delta_{\varepsilon}$  is a restricted Bayes solution with risk bounded by  $\sup_{\theta} R(\theta, \delta_{\varepsilon})$  and the converse also holds.

Our purpose is to apply the restricted Bayes and minimax principles to the problem of estimating  $\theta$  when  $y = \theta + e$  using a quadratic loss  $L(\theta, \delta)$ .

In Section 2 we assume that e has a normal distribution with a known variance. The exact mathematical solution of the restricted Bayes problem in this case is very messy. However, we show that

Minimum Bayes risk = 1 - I(G)

where I(G) denotes the Fisher information for location of the marginal distribution G of y. As G depends on U it follows that the l.f. distribution in P II minimizes I(G) over  $\mathcal{P}_{\varepsilon}$ . This result is used in order to:

- obtain an approximate analytical solution of the restricted optimal problems;
- obtain accurate numerical approximations of the l.f. distribution and of the corresponding optimal estimate.

In Section 3 we exchange the role of prior and error distribution, i.e. we assume that U is Gaussian and that the error model is in a "neighborhood" of a given distribution  $F_0$ , and we modify the restricted Bayes and minimax problems in order to provide a safeguard against deviations from  $F_0$ . It turns out that the approximate solution of the corresponding optimality problem is based on data trimming.

## 2 The case of inaccurate structure function

Let  $y = \theta + e$ . Suppose that the density of e is  $\phi_v(x) = (1/\sqrt{2\pi} v) \exp(-x^2/(2v^2))$ (v known) and that  $\theta$  is distributed according to a structure function U. Let  $f(y \mid \theta)$  denote the density of y for given  $\theta$  and let  $g(y) = f \circ U(y)$  be the marginal density of y where  $f \circ U(y) = \int f(y \mid \theta) dU(\theta)$ . The corresponding cumulative distributions are denoted by  $F(y \mid \theta)$  and  $G(y) = F \circ U(y)$ . Let

$$I(G) = \int \left(\frac{d}{dy}\ln g(y)\right)^2 g(y) \, dy$$

be the Fisher information for location of G.

It is desired to estimate  $\theta$  by an estimate  $\delta$  using the loss  $L(\theta, \delta) = (\theta - \delta)^2$ . Without loss of generality, we restrict our attention to estimators of the form  $\delta(y) = y + \psi(y)$  where  $\psi$  is an absolutely continuous function such that  $E_{\theta}(|\psi'(y)|) < \infty$  and  $E_{\theta}(\cdot)$  denotes the conditional expectation given  $\theta$ .

Lemma 1.

I) 
$$R(\theta, \delta) = v^2 + v^4 E_{\theta}(\psi^2(y) + 2\psi'(y))$$
 for  $\delta(y) = y + v^2 \psi(y)$ .

- II) The Bayes estimator of  $\theta$  is  $\delta_U(y) = y + v^2 g'(y)/g(y)$ .
- III) The minimum Bayes risk is  $r(U, \delta_U) = v^2(1 v^2 I(G))$ .

*Proof.* Consider estimators of the form  $\delta_a(y) = y + a\psi(y)$  where a is an arbitrary constant. We obtain:

$$R(\theta, \delta_a) = E_{\theta}(\delta_a - \theta)^2$$
  
=  $v^2 + a^2 E_{\theta}(\psi^2(y)) + 2aE_{\theta}((y - \theta)\psi(y))$ 

By partial integration  $E_{\theta}((y - \theta)\psi(y)) = E_{\theta}(\psi'(y))v^2$  from which I) follows. Moreover:

$$r(U, \delta_a) = v^2 + a^2 E(\psi^2) + 2a E(\psi') v^2$$

We minimize first on a, the optimal value being

$$a_0 = -v^2 \frac{E(\psi')}{E(\psi^2)}$$
 with  $r(U, \delta_{a_0}) = v^2 - v^4 \frac{E(\psi')^2}{E(\psi^2)}$ .

Then we minimize on  $\psi$  observing that:

$$\frac{E(\psi')^2}{E(\psi^2)} \le \int \left(\frac{g'(y)}{g(y)}\right)^2 g(y) \, dy = I(G)$$

by partial integration and Schwarz's inequality. Hence the Bayes estimator of  $\theta$  is obtained with  $a = v^2$  and  $\psi = g'/g$ . The properties II) and III) follow immediately.

*Remark.* g can be estimated from available collateral data.

#### 2.1 Approximate analytical solution of P I and P II

In order to find a l.f. distribution in  $\mathcal{P}_{\varepsilon}$  one should minimize I(G) on the set

$$\mathscr{R}_{\varepsilon} = \left\{ G \mid G = (1 - \varepsilon)G_0 + \varepsilon K \,, \quad G_0 = F \circ U_0 \,, \quad K = F \circ H \,, \quad H \in \mathscr{H} \right\}.$$

Denote by  $\Theta$  the support of the l.f. contamination  $H_{\varepsilon}$ ; let c be a Lagrange multiplier for the condition  $\int g(y) dy = 1$  and let  $\psi = g'/g$ . By applying variational methods (as in *Huber* [8], p. 82) one obtains the condition

$$c - E_{\theta}(\psi^{2}(y) + 2\psi'(y)) = 0 \quad \text{for} \quad \theta \in \Theta$$
$$\geq 0 \quad \text{for} \quad \theta \notin \Theta.$$

We remark, without surprise, that this coincides with the condition  $R(\theta, \delta) \le m + c_0$  with  $c_0 = c$  in *PI* because y is the minimax estimate with  $m = v^2$ . If K were arbitrary, one would obtain  $c - 2(g'/g)' - (g'/g)^2 = 0$  and this differential equation could be solved for g; unfortunately, the condition that K must be a mixture of normal densities makes the problem much harder.

As the function  $E_{\theta}(\psi^2 + 2v^2\psi')$  is analytic in  $\theta$ , the support  $\Theta$  is a discrete set. A rigorous proof can be found in *Bickel/Collins* [2]. However, we do not know explicit formulae for the masses of  $H_{\varepsilon}$  nor for their abscissae. Therefore, approximate solutions (of approximate optimality problems) are of interest. We consider the following problem (see also *Berger* [1]):

*P I': The approximate restricted Bayes problem.* Minimize  $r(U_0, \delta)$  for  $\delta(y) = y + v^2 \psi(y)$  subject to:

$$\psi^2(y) + 2\psi'(y) \le c_0$$
 for all  $y$ .

This condition is clearly motivated by I) in Lemma 1 and is stronger than the condition in *P I*. On the other hand, we define an extended game  $(\widehat{\mathscr{R}}_{\varepsilon}, \mathscr{D}, \widehat{r})$  where

$$\begin{split} \widehat{\mathscr{R}}_{\varepsilon} &= \{ G \mid G = (1 - \varepsilon)G_0 + \varepsilon K \,, \quad G_0 = F \circ U_0 \,, \\ &\quad K \text{ is an arbitrary contamination} \} \\ \mathscr{D} &= \{ \delta \mid \delta(y) = y + v^2 \psi(y) \} \\ \widehat{r}(G, \psi) &= v^2 + v^4 E_G(\psi^2 + 2\psi') \end{split}$$

and  $E_G(\cdot)$  denotes expectation using the distribution G. We remark that  $\hat{r}(G, \psi)$  coincides with  $r(U, \delta)$  for  $\delta \in \mathcal{D}$  and  $G = F \circ U$  with  $U \in \mathcal{P}_{\varepsilon}$ . Therefore, one can formulate the following problem:

*PII': The approximate restricted minimax problem.* Let  $\varepsilon \in (0, 1)$  be a given number. Find  $\widehat{\psi}_{\varepsilon}$  such that

$$\sup_{\widehat{\mathscr{R}}_{\varepsilon}} \widehat{r}(G, \widehat{\psi}_{\varepsilon}) = \inf_{\mathscr{D}} \sup_{\widehat{\mathscr{R}}_{\varepsilon}} \widehat{r}(G, \psi) \, .$$

By standard arguments, P II' leads to minimization of I(G) over  $\widehat{\mathscr{R}}_{\varepsilon}$  i.e. to the minimum condition:

$$c - 2(g'/g)' - (g'/g)^2 \ge 0.$$

Therefore, PII' is equivalent to PI'. Moreover, assuming  $-\log g_0$  to be convex, the result in *Huber* [8], p. 85 can be used: the l.f. density  $\widehat{g}_{\varepsilon}$  is:

$$\widehat{g}_{\varepsilon}(y) = (1 - \varepsilon)g_0(y_0)e^{d(y - y_0)} \quad \text{for} \quad y \le y_0$$

$$(1 - \varepsilon)g_0(y) \quad \text{for} \quad y_0 < y < y_1$$

$$(1 - \varepsilon)g_0(y_1)e^{-d(y - y_1)} \quad \text{for} \quad y_1 < y$$

where  $d = \sqrt{c}$  is related to  $\varepsilon$  through the condition  $\int \widehat{g}_{\varepsilon}(y) dy = 1$  and  $y_0 < y_1$  are the endpoints of the interval where  $|g'_0/g_0| \le d$ . Finally the approximate restricted minimax estimate is

$$\widehat{\delta}_{\varepsilon}(y) = y + v^2 \widehat{\psi}_{\varepsilon}(y)$$

with  $\widehat{\psi}_{\varepsilon} = \widehat{g}_{\varepsilon}'/\widehat{g}_{\varepsilon}$ . Clearly  $\sup_{\theta} R(\theta, \widehat{\delta}_{\varepsilon}) = v^2 + v^4 d^2$ .

*Example.* Let  $U_0$  be the Gaussian distribution with mean  $\mu$  and variance  $\sigma^2$ ; then  $g_0$  is the density of the Gaussian distribution with mean  $\mu$  and variance  $\tau^2 = \sigma^2 + v^2$ . We obtain:

$$\widehat{\delta}_{\varepsilon}(y) = y + v^2 \max[-d, \min[d, (\mu - y)/\tau^2]]$$

or, more explicitly:

$$\begin{aligned} \widehat{\delta}_{\varepsilon}(y) &= y + v^2 d & \text{for } y < \mu - d\tau^2 \\ y \frac{\sigma^2}{\tau^2} + \mu \frac{v^2}{\tau^2} & \text{for } \mu - d\tau^2 \le y \le \mu + d\tau^2 \\ y - v^2 d & \text{for } y > \mu + d\tau^2 \end{aligned}$$

The constants  $\varepsilon$  and d are related through  $2\phi_{\tau}(d\tau^2)/d + 2\Phi_{\tau}(d\tau^2) = (2-\varepsilon)/(1-\varepsilon)$  where  $\Phi_{\tau}$  denotes the Gaussian distribution with mean 0, variance  $\tau^2$  and density  $\phi_{\tau}$ . We observe that  $\hat{\delta}_{\varepsilon}$  coincides with the *limited* translation rule of Efron/Morris [5], which follows the Bayes rule as closely as possible subject to the condition  $|\delta(y) - y| \le v^2 d$ .

## 2.2 Numerical approximation of the least favorable distribution

The discrete nature of  $\Theta$  suggests the possibility to approximate  $H_{\varepsilon}$  numerically. Indeed, for the case  $u_0(\theta) = \phi_{\sigma}(\theta)$  (the Gaussian density with

mean 0 and variance  $\sigma^2$ ) Marazzi [12] minimizes  $I(\overline{G})$  over the 2n + 2 parameters  $h_1, \ldots, h_n, \theta_1, \ldots, \theta_n, t, b$  of the marginal density

$$\overline{g}(y) = (1-\varepsilon)\phi_{\tau}(y) + \varepsilon \left[\sum_{j=1}^{n} h_{j} \alpha(y;\theta_{j}) + h \int_{0}^{\infty} \alpha(y;t+\theta) e^{-b\theta} b \, d\theta\right]$$

where  $\alpha(y;\theta) = \phi_v(y+\theta) + \phi_v(y-\theta)$ ,  $\sum h_j + h = 0.5$  and  $\tau^2 = v^2 + \sigma^2$ . By choosing *n* sufficiently large, the least favorable marginal density  $g_{\varepsilon} = (1-\varepsilon)\phi_{\tau} + \varepsilon\phi_v \circ H_{\varepsilon}$  may be approximated as precisely as desired by functions of this form. Note that  $\overline{g}$  has been constructed so that the asymptotic behaviours for large arguments of  $\overline{g}'/\overline{g}$  and of the risk function of  $\overline{\delta} = y + v^2 \overline{g}'/\overline{g}$  coincide with the corresponding behaviours of the analytical approximation of Section 2.1.

In Figure 1 (taken from *Marazzi* [12]) the functions  $-\widehat{g}_{\varepsilon}'/\widehat{g}_{\varepsilon}$  and  $-\overline{g}_{\varepsilon}'/\overline{g}_{\varepsilon}$  (obtained by minimizing  $I(\overline{G})$ ) are drawn together with the corresponding risk functions. The numerical approximation mimics the oscillatory behaviour of the optimal rule for low values of y and replaces the oscillations by a simpler curve for those values of y which do not appreciably affect the interesting mean risks  $r(U_0, \overline{\delta}_{\varepsilon})$  and  $r(\overline{H}_{\varepsilon}, \overline{\delta}_{\varepsilon})$ .

Some of the numerical results are indicated in Table 1 where the value of  $r(\overline{H}_{\varepsilon}, \overline{\delta}_{\varepsilon})$  is an approximation for  $\sup_{\theta} R(\theta, \overline{\delta}_{\varepsilon})$ . Clearly

$$I(\widehat{G}_{\varepsilon}) \leq \min_{\mathscr{P}_{\varepsilon}} I(G) \leq I(\overline{G}_{\varepsilon})$$

and, as the lower bound is numerically close to the upper bound, it can be concluded that the analytical (and the numerical) approximation is nearly optimal.

## 3 The case of inaccurate error distribution

Let  $y = \theta + e$ . Suppose that the structure function U is the Gaussian distribution with mean  $\mu$  and variance  $\sigma^2$  and note the distribution of e by F with density f. We define the *a priori mean squared loss function* of an estimator  $\delta$  of  $\theta$  as

$$l(e, \delta) = \int L(\theta, \delta(\theta + e)) dU(\theta)$$

3	0.1	0.2	0.3	0.4	0.5	0.1	0.2	0.3	0.4	0.5
$(v^2, \sigma^2)$			(1,0)					(1,1)		
$I(\widehat{g}_{\varepsilon})$	0.671	0.489	0.354	0.250	0.169	0.336	0.244	0.177	0.125	0.084
$r(U_0,\widehat{\delta}_{\varepsilon})$	0.110	0.203	0.293	0.381	0.472	0.555	0.602	0.646	0.691	0.736
$\sup R(\theta, \widehat{\delta}_{\varepsilon})$	2.230	1.742	1.469	1.302	1.190	1.650	1.371	1.234	1.151	1.095
$I(\overline{g}_{\varepsilon})$	0.697	0.520	0.385	0.279	0.193	0.340	0.250	0.182	0.130	0.088
$r(U_0,\overline{\delta}_{\varepsilon})$	0.092	0.176	0.257	0.340	0.425	0.552	0.597	0.640	0.684	0.728
$r(\overline{H}_{\varepsilon},\overline{\delta}_{\varepsilon})$	2.194	1.698	1.447	1.294	1.190	1.633	1.364	1.231	1.149	1.095
n	3	3	4	4	4	4	5	5	5	5

Table 1. Numerical results



Figure 1. Minimax functions  $-\overline{g}_{\varepsilon}'/\overline{g}_{\varepsilon}$ ,  $-\widehat{g}_{\varepsilon}/\widehat{g}_{\varepsilon}$  and corresponding risk functions (for  $\varepsilon = 0.1$ ,  $v^2 = 1$ ,  $\sigma^2 = 0$ , n = 3)

and use  $L(\theta, \delta) = (\theta - \delta)^2$ . The mean (Bayes) risk is then

$$r(F,\delta) = \int l(e,\delta) \, dF(e) \, .$$

In this section it will be convenient to consider the mean risk as a function of  $\delta$  and F. As in Section 2 let  $g(y) = f \circ U(y) = \int f(y - \theta) dU(\theta)$  and I(G) be the Fisher information for location of G.

# Lemma 2.

I)  $l(e, \delta) = \sigma^2 + \sigma^4 \int (\psi^2(y) + 2\psi'(y)) dU(\theta)$  for  $\delta(y) = \mu - \sigma^2 \psi(y)$ .

II) The Bayes estimator of  $\theta$  is  $\delta_F(y) = \mu - \sigma^2 g'(y)/g(y)$ .

III) The minimum Bayes risk is  $r(F, \delta_F) = \sigma^2 (1 - \sigma^2 I(G))$ .

*Proof.* Similar to the proof of Lemma 1 in Section 2.

# 3.1 Modified restricted optimality principles

In Section 2 the cause of an "outlying" value of y was assigned to an outlying value of  $\theta$  and it was appropriate to obtain robustness by bounding  $R(\theta, \delta)$ . It is clearly impossible to define a sensible robust procedure based on the single observation y without assuming one of its two components,  $\theta$  or e, to be correct. Yet, it does not seem unreasonable to treat the two components of y in a similar way. Therefore, if the cause of an outlying value of y is now assigned to a bad value of e it may be appropriate to obtain robustness by bounding  $l(e, \delta)$ . The following approach simply paraphrases the previous section by exchanging the roles of the structure function and the error distribution.

Let  $F_0$  be a given error distribution. The goal is to find an optimal estimator of  $\theta$  that utilizes the information contained in  $F_0$  but, at the same time, provide a safeguard in case this information is incorrect. Consider  $\delta^0(y) \equiv \mu$ , the prior mean of  $\theta$ ; this estimator does not use  $F_0$  at all and its maximum a priori mean squared loss  $\sup_e l(e, \delta^0) \equiv l(e, \delta^0) \equiv \sigma^2$  is the smallest possible value for the maximum of the a priori mean squared loss function. But we might be willing to tolerate a somewhat larger maximum a priori mean loss if there results a substantial decrease in the mean risk when  $F_0$  is correct. This leads to the following problem:

*Pi*: The modified restricted Bayes problem. Minimize  $r(F_0, \delta)$  subject to the condition

$$l(e, \delta) \le \sigma^2 + c_0$$
 for all  $e$   $(c_0 > 0)$ .

The Hodges & Lehmann theory for PI and PII can obviously be applied by exchanging the prior and error distributions. In particular one needs to consider the sets

$$\mathscr{P}_{\varepsilon} = \{ F \mid F = (1 - \varepsilon)F_0 + \varepsilon H, \quad H \in \mathscr{H} \}$$

where  $\mathscr{H} = \{$ all distributions $\}$  and

$$\mathscr{R}_{\varepsilon} = \{ G \mid G = (1 - \varepsilon)G_0 + \varepsilon K, \quad G_0 = F_0 \circ U, \quad K = H \circ U, \quad H \in \mathscr{H} \}$$

where  $\varepsilon \in (0, 1)$ . There is an equivalent minimax problem:

*P ii: The modified restricted minimax problem.* Minimize the maximum Bayes risk over all error distributions  $F \in \mathcal{P}_{\varepsilon}$ .

From Lemma 2 it follows that the l.f. distribution  $H_{\varepsilon}$  in *P* ii minimizes I(G) over  $\mathscr{P}_{\varepsilon}$  and  $\mathscr{R}_{\varepsilon}$ . With the aid of a Lagrange multiplier for  $\int g(y) dy = 1$  we obtain the minimum condition

$$c - \int (\psi^2(y) + 2\psi'(y))f(y - \theta) dU(\theta) = 0$$
 for  $y \in \Gamma$ 

where  $\psi = g'/g$  and  $\Gamma$  is the support of  $H_{\varepsilon}$ . The equality sign is replaced by  $\geq$  for  $y \notin \Gamma$ .

A numerical procedure similar to the one described in Section 2.2 is applicable to the determination of  $H_{\varepsilon}$  but we may be satisfied with the l.f. marginal distribution  $\hat{G}_{\varepsilon}$  in the extended class

$$\widehat{\mathscr{R}}_{\varepsilon} = \{ G \mid G = (1 - \varepsilon)G_0 + \varepsilon K, \quad G_0 = F_0 \circ U, \quad K \text{ arbitrary} \}.$$

The form of  $\hat{g}_{\varepsilon}$  is the same as in Section 2.1 and the approximate restricted minimax estimate is

$$\widehat{\delta}_{\varepsilon}(y) = \mu - \sigma^2 \widehat{\psi}_{\varepsilon}(y)$$

where  $\widehat{\psi}_{\varepsilon} = \widehat{g}_{\varepsilon}'/\widehat{g}_{\varepsilon}$ . Clearly  $\sup_{e} l(e, \widehat{\delta}_{\varepsilon}) = \sigma^{2} + \sigma^{4} d^{2}$ .

# 4 Extensions and open problems

The method has been extended by Marazzi [12] to the linear model

$$y = X\vartheta + e$$

where y is an *n*-vector of observations,  $\vartheta$  a *p*-vector of parameters, X an  $n \times p$  matrix of constants and *e* an *n*-vector of errors. Two cases have been considered: (a) the distribution of *e* is the *n*-variate Gaussian distribution and the *p*-variate structure function is not exactly known; (b) the structure function is a *p*-variate Gaussian distribution and the specified *n*-variate error distribution  $F_0$  is affected by contamination.

For example, in case (b) with p = 1 and  $X = (1, ..., 1)^T$ , if  $F_0$  is the *n*-variate Gaussian distribution with mean vector O and covariance matrix  $v^2 I$  one obtains an approximate restricted minimax estimate of the form

$$\widehat{\delta}_{\varepsilon}(\mathbf{y}) = \mu - \sigma^2 \max[-d_n, \min[d_n, (\mu - \overline{\mathbf{y}})/\tau^2]]$$

where  $\overline{y}$  is the arithmetic mean of the components of y,  $d_n$  is an appropriate constant,  $\mu = E(\vartheta)$ ,  $\sigma^2 = \text{Var}(\vartheta)$  and  $\tau^2 = \sigma^2 + v^2/n$ .

The estimate  $\hat{\delta}_{\varepsilon}(\mathbf{y})$  is based on the assumption that the components of  $\mathbf{e}$  are independent and identically distributed with probability  $(1 - \varepsilon)$ ; however,  $\mathbf{e}$  comes from an arbitrary multivariate contamination with probability  $\varepsilon$ . A different model assumes that the distribution of  $\mathbf{e} = (e_1, \dots, e_n)^T$  is of the form

$$F(e) = F_1(e_1) \cdot F_2(e_2) \cdot \cdots \cdot F_n(e_n)$$

where each factor  $F_i$  is a mixture of a given univariate distribution and an arbitrary univariate contamination. The application of the restricted Bayes and minimax principles to this situation is still an open problem.

The crucial identity of Section 2 allowing to relate the minimum Bayes risk to the Fisher information is  $E_{\theta}((y - \theta)\psi(y)) = v^2 E(\psi'(y))$ . This identity can be generalized to the continuous exponential family

$$f(y \mid \theta) = \exp(\theta y - \gamma(\theta))\beta(y)$$

with support  $\mathbb{R} = (-\infty, \infty)$ . If the support is a bounded interval we need a supplementary condition (see *Hudson* [9]). Indeed, with  $s(y) = -\beta'(y)/\beta(y)$ , we obtain

$$E_{\theta}((s(y) - \theta)\psi(y)) = E_{\theta}(\psi'(y))$$

for any absolutely continuous function  $\psi$  on  $\mathbb{R}$  such that  $E_{\theta}(|\psi'(y)|) < \infty$ . The Bayes estimator of  $\theta$  with respect to a prior distribution U is  $\delta_U(y) = s(y) + g'(y)/g(y)$  and the minimum Bayes risk is  $r(\delta_U, U) = E_G(s'(y)) - I(G)$ . A similar extension of the results of Section 3 is also possible.

Therefore, in order to find a l.f. distribution in a (weakly compact and convex) set of prior distributions  $\mathscr{P}$  one has to minimize the functional  $J(G) = I(G) - E_G(s')$  on  $\mathscr{R} = \{G \mid G(y) = \int F(y \mid \theta) dU(\theta), \quad U \in \mathscr{P}\}$ . Again we may consider  $\mathscr{P} = \mathscr{P}_{\varepsilon}$  in which case we note  $\mathscr{R}$  by  $\mathscr{R}_{\varepsilon}$ . Moreover, if we allow G to belong to the extended set  $\widehat{\mathscr{R}}_{\varepsilon}$ , we obtain the condition

$$c - s' - 2(g'/g)' - (g'/g)^2 = 0$$

on the set of y-values where g can be freely varied. Writing  $z(y) = \sqrt{g(y)}$  the equation becomes

$$z''(y) + \frac{1}{4}[s'(y) - c]z(y) = 0.$$

Beyond this point the problem remains open.

Alfio Marazzi Institut universitaire de médecine sociale et préventive Bugnon 17, 1004 Lausanne

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#### Summary

The restricted Bayes and minimax principles are used in order to derive robust credibility estimates of a risk parameter  $\theta$  in the very simple case where the claim is the sum of  $\theta$  and an "error term". Two examples are considered: (a) the error distribution is Gaussian and the structure function is not exactly known; (b) the structure function is Gaussian and the error model is not precise. Approximate analytical and numerical solutions as well as possible extensions are discussed.

## Zusammenfassung

Eingeschränkte Bayes- und Minimaxprinzipien werden angewandt, um robuste Credibility Schätzungen eines Risikoparameters  $\theta$  abzuleiten, dies im ganz einfachen Falle, wo der Schadenbetrag der Summe von  $\theta$  und einem "Fehlerwert" gleich gesetzt ist. Zwei Beispiele werden betrachtet: (a) die Fehlerverteilung ist nach Gauss und die Strukturfunktion nicht genau bekannt; (b) die Strukturfunktion ist nach Gauss und das Fehlermodell ungenau. Annähernde analytische und numerische Lösungen sowie mögliche Erweiterungen werden beschrieben.

## Résumé

Les critères restreints de Bayes et minimax sont utilisés pour développer des estimateurs de crédibilité robustes d'un paramètre de risque  $\theta$  dans le cas simple où le sinistre est la somme de  $\theta$  et d'une "erreur". Deux exemples sont considérés: (a) la distribution de l'erreur est gaussienne et la fonction de structure n'est connue qu'approximativement; (b) la fonction de structure est gaussienne et le modèle d'erreur est imprécis. Des solutions approximatives analytiques et numériques ainsi que des extensions possibles sont décrites.