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Pricing Financial Contracts with Indexed Homogeneous Payoff

I Introduction

Consider a financial contract for a single payment to be made in the future. The amount of the payment depends on the time of the payment and the prices of two stocks or assets. For $j = 1, 2$, and $t \geq 0$, let $S_j(t)$ denote the price of stock j at time t. If the payment is made at time t, the amount is

$$P(t, S_1(t), S_2(t)) . \tag{1.1}$$

We assume a *risk-neutral world*. The force of interest is constant and it is denoted by δ. The current value of the contract, if the payment is to be made at time t, is the expectation

$$E[e^{-\delta t} P(t, S_1(t), S_2(t))] . \tag{1.2}$$

If the timing of the payment is controlled by the payee (who has the role of a creditor), then the price of the contract is

$$\sup_T E[e^{-\delta T} P(T, S_1(T), S_2(T))] , \tag{1.3}$$

where T denotes stopping times (exercise strategies). On the other hand, if the timing of the payment is controlled by the payer (debitor), then the price of the contract its

$$\inf_T E[e^{-\delta T} P(T, S_1(T), S_2(T))] . \tag{1.4}$$

Motivated by Shepp and Shiryaev (1993b), we shall limit ourselves to payment functions of the form

$$P(t, s_1, s_2) = e^{gt} \Pi(s_1, s_2) , \tag{1.5}$$

where g is a constant, and Π is a positively homogeneous function of degree one. In particular, we study

$$P(t, s_1, s_2) = e^{gt} \operatorname{Max}(s_1, s_2) , \tag{1.6}$$

where $g < 0$ and the time of the payment is chosen by the payee, and the dual problem

$$P(t, s_1, s_2) = e^{gt} \operatorname{Min}(s_1, s_2), \tag{1.7}$$

where $g > 0$ and the time of the payment is determined by the payer. Note that the factor e^{gt} can be interpreted as an index; its instantaneous rate of change is negative in (1.6) and positive in (1.7).

The functions Π in (1.6) and (1.7) are symmetric in their arguments. This symmetry is exploited in determining the contract-pricing formulas. Another interesting example of such functions is

$$\Pi(s_1, s_2) = |s_1 - s_2|,$$

which is the payoff function of the *symmetric Margrabe option*.

The term "Russian option" was coined by Shepp and Shiryaev (1993a) to describe a perpetual (American) option on a stock whose payoff is the historical maximum value of the stock prices up till the time of option exercise. Here we study an indexed version of the Russian option pricing problem. We take this opportunity to thank Dr. L.A. Shepp for encouraging us to investigate the dual Russian option problem.

2 The Model

We consider two non-dividend-paying stocks. For $j = 1, 2$, and $t \geq 0$, let $S_j(t)$ denote the price of stock j at time t, and

$$X_j(t) = \ln[S_j(t)/S_j(0)] \tag{2.1}$$

be its cumulative force of return. We assume that $\{(X_1(t), X_2(t)); t \geq 0\}$ is a two-dimensional Wiener process with parameters $\mu_1, \mu_2, \sigma_1, \sigma_2$ and ϱ, $|\varrho| < 1$, so that the joint moment generating function of $X_1(t)$ and $X_2(t)$ is

$$E[e^{k_1 X_1(t) + k_2 X_2(t)}] = \exp\left\{t\left(k_1\mu_1 + k_2\mu_2 + \frac{1}{2}k_1^2\sigma_1^2 + \varrho k_1 k_2 \sigma_1 \sigma_2 + \frac{1}{2}k_2^2\sigma_2^2\right)\right]. \tag{2.2}$$

We further assume that these parameters are the specifications of the *risk-neutral probability measure* (which is normally different from the *physical probability* *measure*). Then the *price* of a random payment is calculated as the expectation of its discounted value. In particular, for $j = 1, 2$,

$$S_j(0) = e^{-\delta t} E[S_j(t)], \tag{2.3}$$

from which it follows that

$$\mu_j = \delta - \frac{1}{2}\sigma_j^2. \tag{2.4}$$

For notational convenience, we shall write S_j for $S_j(0)$.

3 Two Expected Discounted Values

For contracts with positively homogeneous payment functions Π, reasonable exercise strategies are stopping times of the form

$$T_{b,c} = \inf\left\{t \mid \frac{S_1(t)}{S_2(t)} = b \quad \text{or} \quad \frac{S_1(t)}{S_2(t)} = c\right\}, \tag{3.1}$$

with $0 < b < S_1/S_2 < c < \infty$. Let 1_A denote the indicator random variable of an event A and g be a constant. Consider the following two expectations of discounted contingent payments of amount

$$e^{gT_{b,c}} S_2(T_{b,c}) \tag{3.2}$$

(the indexed price of stock 2 at time $T_{b,c}$),

$$\begin{aligned} \beta &= \beta(S_1, S_2; b, c) \\ &= E[e^{-(\delta-g)T_{b,c}} S_2(T_{b,c}) 1_{\{S_1(T_{b,c})=bS_2(T_{b,c})\}}] \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} \gamma &= \gamma(S_1, S_2; b, c) \\ &= E[e^{-(\delta-g)T_{b,c}} S_2(T_{b,c}) 1_{\{S_1(T_{b,c})=cS_2(T_{b,c})\}}]. \end{aligned} \tag{3.4}$$

We shall obtain explicit expressions for β and γ by a martingale argument. The idea is to determine constants θ for which the process

$$\{e^{-(\delta-g)t} S_1(t)^\theta S_2(t)^{1-\theta}; t \geq 0\} \tag{3.5}$$

is a martingale. This leads to the condition

$$e^{-(\delta-g)t} E[e^{\theta X_1(t)+(1-\theta)X_2(t)}] = 1. \tag{3.6}$$

Using (2.2) we obtain a quadratic equation for θ:

$$-(\delta - g) + \theta\mu_1 + (1-\theta)\mu_2 + \frac{1}{2}\theta^2\sigma_1^2 + \varrho\theta(1-\theta)\sigma_1\sigma_2 + \frac{1}{2}(1-\theta)^2\sigma_2^2 = 0\,. \tag{3.7}$$

With the notation

$$\nu^2 = \mathrm{Var}[X_1(1) - X_2(1)] = \sigma_1^2 - 2\varrho\sigma_1\sigma_2 + \sigma_2^2 \tag{3.8}$$

and applying (2.4), we can simplify equation (3.7) to

$$g - \frac{1}{2}\nu^2\theta(1-\theta) = 0\,. \tag{3.9}$$

Let

$$\triangle = \sqrt{1 - \frac{8g}{\nu^2}}\,; \tag{3.10}$$

then the roots of (3.9) are

$$\theta_0 = \frac{1}{2} - \frac{1}{2}\triangle \tag{3.11}$$

and

$$\theta_1 = \frac{1}{2} + \frac{1}{2}\triangle\,. \tag{3.12}$$

To classify them we distinguish five cases according to the values of g:

$g < 0$	$\triangle > 1$	$\theta_0 < 0, \quad 1 < \theta_1$
$g = 0$	$\triangle = 1$	$\theta_0 = 0, \quad 1 = \theta_1$
$0 < g < \nu^2/8$	$0 < \triangle < 1$	$0 < \theta_0 < \theta_1 < 1$
$g = \nu^2/8$	$\triangle = 0$	$\theta_0 = \frac{1}{2} = \theta_1$
$g > \nu^2/8$	$\triangle = i\kappa, \quad \kappa > 0$	$\theta_0 = \frac{1}{2} - \frac{1}{2}i\kappa$, $\theta_1 = \frac{1}{2} + \frac{1}{2}i\kappa$

Notation. For notational simplicity we may write the symbol θ, without any subscript, in the following. In such situations, the reader will find that the formulas are valid whether θ is interpreted as θ_0 or θ_1. The reader may also find it instructive to express such formulas in terms of $\triangle$ or κ.

Let us exclude the case $g = \nu^2/8$ for the moment. Then, for $j = 0, 1$, stopping the martingale

$$\{e^{-(\delta-g)t} S_1(t)^{\theta_j} S_2(t)^{1-\theta_j}; t \geq 0\} \tag{3.13}$$

at time $T_{b,c}$ and applying the optional sampling theorem yields the equation

$$S_1(0)^{\theta_j} S_2(0)^{1-\theta_j} = b^{\theta_j}\beta + c^{\theta_j}\gamma\,. \tag{3.14}$$

With the definition

$$D = b^{1-\theta}c^{\theta} - b^{\theta}c^{1-\theta}\,, \tag{3.15}$$

the solution of the two linear equations (3.14), $j = 0, 1$, is

$$\beta = \frac{1}{D}[S_1^{1-\theta}(cS_2)^{\theta} - S_1^{\theta}(cS_2)^{1-\theta}] \tag{3.16}$$

and

$$\gamma = \frac{1}{D}[S_1^{\theta}(bS_2)^{1-\theta} - S_1^{1-\theta}(bS_2)^{\theta}]\,. \tag{3.17}$$

In the limiting case $g = \nu^2/8$, we apply the optional sampling theorem to the martingales

$$\{e^{-(\delta-g)t}\sqrt{S_1(t)S_2(t)}\,; t \geq 0\} \tag{3.18}$$

and

$$\{e^{-(\delta-g)t}\sqrt{S_1(t)S_2(t)}\,\ln[S_1(t)/S_2(t)]; t \geq 0\} \tag{3.19}$$

to get the equations

$$\sqrt{S_1S_2} = \sqrt{b}\beta + \sqrt{c}\gamma$$

and

$$\sqrt{S_1S_2}\,\ln(S_1/S_2) = \sqrt{b}\ln(b)\beta + \sqrt{c}\ln(c)\gamma\,,$$

whose solution is

$$\beta = \sqrt{S_1S_2/b}\,\frac{\ln(S_1/cS_2)}{\ln(b/c)} \tag{3.20}$$

and

$$\gamma = \sqrt{S_1S_2/c}\,\frac{\ln(S_1/bS_2)}{\ln(c/b)}\,. \tag{3.21}$$

If $\triangle$ is an imaginary number, $\triangle = i\kappa$, $\kappa > 0$, the expectations (3.3) and (3.4) are finite provided that

$$c/b < e^{2\pi/\kappa}\,. \tag{3.22}$$

It is instructive to write (3.16) and (3.17) in terms of real-valued functions. We find that

$$\beta = \sqrt{S_1 S_2/b}\,\frac{\sin[\kappa \ln(cS_2/S_1)/2]}{\sin[\kappa \ln(c/b)/2]}\,, \tag{3.23}$$

and

$$\gamma = \sqrt{S_1 S_2/c}\,\frac{\sin[\kappa \ln(S_1/bS_2)/2]}{\sin[\kappa \ln(c/b)/2]}\,. \tag{3.24}$$

Formulas (3.23) and (3.24) show that the expectations (3.3) and (3.4) tend to ∞ as $\ln(c/b)$ tends to $2\pi/\kappa$. In the following we shall tacitly assume (3.22) if $\triangle$ is imaginary. Note that formulas (3.20) and (3.21) are the limits, as κ tends to 0, of (3.23) and (3.24).

Remark. Many of the results in this paper can be extended to the case where each stock pays a continuous stream of dividends, at a rate proportional to its price, i.e., for $j = 1, 2$, there is a positive constant α_j such that the amount of dividends paid by stock j between time t and $t + dt$ is $S_j(t)\alpha_j\,dt$. Then condition (2.4) becomes

$$\mu_j = \delta - \alpha_j - \frac{1}{2}\sigma_j^2\,. \tag{3.25}$$

Although we shall not discuss this generalization, we note that, for some of the more attractive results, the relationship

$$\theta_0 + \theta_1 = 1 \tag{3.26}$$

is crucial; this is equivalent to the assumption that

$$\alpha_1 = \alpha_2\,. \tag{3.27}$$

4 Pricing Contracts with Indexed Homogeneous Payoff

In the following, $\Pi(s_1, s_2)$ denotes a positively homogeneous function of degree one. Thus

$$\Pi(s_1, s_2) = s_2 \pi\left(\frac{s_1}{s_2}\right),$$

where

$$\pi(s) = \Pi(s, 1). \tag{4.1}$$

We now consider a financial contract with a single indexed payment. If the payment is made at time t, its amount is

$$e^{gt}\Pi(S_1(t), S_2(t)). \tag{4.2}$$

According to the sign of the constant g, two cases have to be distinguished. For $g < 0$, the timing of the payment (4.2) is to be controlled by the payee (creditor), and the current price of the contract is the supremum, over all stopping times T, of

$$E[e^{-(\delta-g)T}\Pi(S_1(T), S_2(T))]. \tag{4.3}$$

For $g > 0$, the timing of the payment (4.2) is controlled by the payer (debitor), and the current price of the contract is the infimum, over all stopping times T, of (4.3).

Since Π is homogeneous of degree 1, we can typically limit ourselves to exercise strategies of the type $T_{b,c}$, with $0 < b < S_1/S_2 < c < \infty$. The current value of such a strategy is

$$\begin{aligned} V(S_1, S_2; b, c) &= E[e^{-(\delta-g)T_{b,c}}\Pi(S_1(T_{b,c}), S_2(T_{b,c}))] \\ &= \pi(b)\beta(S_1, S_2; b, c) + \pi(c)\gamma(S_1, S_2; b, c). \end{aligned} \tag{4.4}$$

The optimal contract-exercise ratios $b = \widetilde{b}$ and $c = \widetilde{c}$ are obtained from the first order conditions

$$V_b(S_1, S_2; \widetilde{b}, \widetilde{c}) = 0 \tag{4.5}$$

and

$$V_c(S_1, S_2; \widetilde{b}, \widetilde{c}) = 0, \tag{4.6}$$

where the subscripts denote partial differentiation. Here we assume that $\widetilde{b} > 0$ and $\widetilde{c} < \infty$, and we shall see that $\widetilde{b}$ and $\widetilde{c}$ do not depend on S_1 or S_2. For $\widetilde{b} < S_1/S_2 < \widetilde{c}$, the current price of the contract is

$$V(S_1, S_2; \widetilde{b}, \widetilde{c}) = \pi(\widetilde{b})\beta(S_1, S_2; \widetilde{b}, \widetilde{c}) + \pi(\widetilde{c})\gamma(S_1, S_2; \widetilde{b}, \widetilde{c})\,. \tag{4.7}$$

5 Smooth Pasting Conditions

The first order conditions (4.5) and (4.6) can be written in an alternative form. First we gather from (4.4) that

$$\begin{aligned} V_b(S_1, S_2; b, c) = \pi'(b)\beta(S_1, S_2; b, c) + \pi(b)\beta_b(S_1, S_2; b, c) \\ + \pi(c)\gamma_b(S_1, S_2; b, c)\,. \end{aligned} \tag{5.1}$$

At the end of this section we shall show that

$$\beta_b(S_1, S_2; b, c) + \beta_{S_1}(b, 1; b, c)\beta(S_1, S_2; b, c) = 0 \tag{5.2}$$

and

$$\gamma_b(S_1, S_2; b, c) + \gamma_{S_1}(b, 1; b, c)\beta(S_1, S_2; b, c) = 0\,. \tag{5.3}$$

Substituting (5.2) and (5.3) in (5.1) yields

$$\begin{aligned} V_b(S_1, S_2; b, c) = \beta(S_1, S_2; b, c)[\pi'(b) - \pi(b)\beta_{S_1}(b, 1; b, c) \\ - \pi(c)\gamma_{S_1}(b, 1; b, c)]\,. \end{aligned} \tag{5.4}$$

Similarly, we obtain

$$\begin{aligned} V_c(S_1, S_2; b, c) = \gamma(S_1, S_2; b, c)[\pi'(c) - \pi(b)\beta_{S_1}(c, 1; b, c) \\ - \pi(c)\gamma_{S_1}(c, 1; b, c)]\,. \end{aligned} \tag{5.5}$$

Since $\beta(S_1, S_2; b, c) > 0$ and $\gamma(S_1, S_2; b, c) > 0$, the first order conditions (4.5) and (4.6) are equivalent to

$$\pi(\widetilde{b})\beta_{S_1}(\widetilde{b}, 1; \widetilde{b}, \widetilde{c}) + \pi(\widetilde{c})\gamma_{S_1}(\widetilde{b}, 1; \widetilde{b}, \widetilde{c}) = \pi'(\widetilde{b}) \tag{5.6}$$

and

$$\pi(\widetilde{b})\beta_{S_1}(\widetilde{c}, 1; \widetilde{b}, \widetilde{c}) + \pi(\widetilde{c})\gamma_{S_1}(\widetilde{c}, 1; \widetilde{b}, \widetilde{c}) = \pi'(\widetilde{c})\,, \tag{5.7}$$

which may be written as

$$V_{S_1}(\widetilde{b}S_2, S_2; \widetilde{b}, \widetilde{c}) = \Pi_{S_1}(\widetilde{b}S_2, S_2) \tag{5.8}$$

and

$$V_{S_1}(\widetilde{c}S_2, S_2; \widetilde{b}, \widetilde{c}) = \Pi_{S_1}(\widetilde{c}S_2, S_2)\,. \tag{5.9}$$

Since

$$V(bS_2, S_2; b, c) = \Pi(bS_2, S_2) \tag{5.10}$$

and

$$V(cS_2, S_2; b, c) = \Pi(cS_2, S_2)\,, \tag{5.11}$$

conditions (5.8) and (5.9) are equivalent to

$$V_{S_2}(\widetilde{b}S_2, S_2; \widetilde{b}, \widetilde{c}) = \Pi_{S_2}(\widetilde{b}S_2, S_2) \tag{5.12}$$

and

$$V_{S_2}(\widetilde{c}S_2, S_2; \widetilde{b}, \widetilde{c}) = \Pi_{S_2}(\widetilde{c}S_2, S_2)\,. \tag{5.13}$$

Formulas (5.8), (5.9), (5.12) and (5.13) are known as the *smooth pasting conditions*. They show that the gradients of the functions $V(\cdot, \cdot; \widetilde{b}, \widetilde{c})$ and $\Pi(\cdot, \cdot)$ coincide on the optimal contract-exercise boundaries $S_1 = \widetilde{b}S_2$ and $S_1 = \widetilde{c}S_2$.
To end this section, we derive (5.2) and (5.3). For x with $b + x < S_1/S_2 < c$, we have the probabilistic identities

$$\beta(S_1, S_2; b, c) = \beta(S_1, S_2; b + x, c)\beta(b + x, 1; b, c) \tag{5.14}$$

and

$$\gamma(S_1, S_2; b, c) = \gamma(S_1, S_2; b + x, c) + \beta(S_1, S_2; b + x, c)\gamma(b + x, 1; b, c)\,. \tag{5.15}$$

Differentiating these equations with respect to x and setting $x = 0$ yields (5.2) and (5.3).

6 The Optimal Exercise Ratios

Consider

$$\Pi(s_1, s_2) = \operatorname{Max}(s_1, s_2)\,. \tag{6.1}$$

For such a function, obviously,

$$\Pi_{S_1}(bS_2, S_2) = 0\,, \qquad 0 < b < 1\,, \tag{6.2}$$

and

$$\Pi_{S_2}(cS_2, S_2) = 0\,, \qquad 1 < c < \infty\,. \tag{6.3}$$

We now derive a condition for the optimal exercise ratios $\widetilde{b}$ and $\widetilde{c}$ for functions Π satisfying (6.2) and (6.3). Here the smooth pasting conditions (5.8) and (5.13) are simply

$$V_{S_1}(\widetilde{b}S_2, S_2; \widetilde{b}, \widetilde{c}) = 0 \tag{6.4}$$

and

$$V_{S_2}(\widetilde{c}S_2, S_2; \widetilde{b}, \widetilde{c}) = 0\,. \tag{6.5}$$

From (4.4), (3.15), (3.16) and (3.17) it follows that

$$V(S_1, S_2; b, c) = w_1(b,c)S_1^{\theta}S_2^{1-\theta} + w_2(b,c)S_1^{1-\theta}S_2^{\theta}\,, \tag{6.6}$$

where the coefficients $w_1(b,c)$ and $w_2(b,c)$ do not depend on S_1 or S_2. Thus conditions (6.4) and (6.5) can be written as

$$w_1(\widetilde{b},\widetilde{c})\theta\widetilde{b}^{\,\theta-1} = -w_2(\widetilde{b},\widetilde{c})(1-\theta)\widetilde{b}^{\,-\theta} \tag{6.7}$$

and

$$w_1(\widetilde{b},\widetilde{c})(1-\theta)\widetilde{c}^{\,\theta} = -w_2(\widetilde{b},\widetilde{c})\theta\widetilde{c}^{\,1-\theta}\,. \tag{6.8}$$

Dividing (6.7) by (6.8) to cancel both $w_1(\widetilde{b},\widetilde{c})$ and $w_2(\widetilde{b},\widetilde{c})$, and solving for $\widetilde{c}/\widetilde{b}$ yields the desired condition

$$\frac{\widetilde{c}}{\widetilde{b}} = \left(\frac{\theta}{\theta-1}\right)^{2/(2\theta-1)} = \left(\frac{\triangle+1}{\triangle-1}\right)^{2/\triangle}\,. \tag{6.9}$$

In the case of (6.1), we are interested in negative g. As g decreases from 0 to $-\infty$, $\triangle$ increases from 1 to ∞, and the ratio $\widetilde{c}/\widetilde{b}$ decreases from ∞ to 1.
Next, we consider functions Π, such as

$$\Pi(s_1, s_2) = \mathrm{Min}(s_1, s_2)\,, \tag{6.10}$$

which satisfy

$$\Pi_{S_2}(bS_2, S_2) = 0\,, \qquad 0 < b < 1\,, \tag{6.11}$$

and

$$\Pi_{S_1}(cS_2, S_2) = 0\,, \qquad 1 < c < \infty\,. \tag{6.12}$$

Here the smooth pasting conditions (5.9) and (5.12) are

$$V_{S_1}(\widetilde{c}\widetilde{S}_2, S_2; \widetilde{b}, \widetilde{c}) = 0 \tag{6.13}$$

and

$$V_{S_2}(\widetilde{b}S_2, S_2; \widetilde{b}, \widetilde{c}) = 0\,. \tag{6.14}$$

As compared to the case above, the role of $\widetilde{b}$ and $\widetilde{c}$ is interchanged. Thus the desired condition is

$$\frac{\widetilde{c}}{\widetilde{b}} = \left(\frac{\theta - 1}{\theta}\right)^{2/(2\theta-1)} = \left(\frac{\triangle - 1}{\triangle + 1}\right)^{2/\triangle}\,. \tag{6.15}$$

In the case of (6.10) we are interested in positive g. For $0 < g < \nu^2/8$, we have $0 < \triangle < 1$; hence condition (6.15) cannot be satisfied. For $g > \nu^2/8$, we have $\triangle = i\kappa$, $\kappa > 0$; ironically, the situation is more promising. With the definition

$$\varphi = \operatorname{arccot}(\kappa)\,, \qquad 0 < \varphi < \pi/2\,, \tag{6.16}$$

we obtain

$$\triangle - 1 = -1 + i\kappa = \sqrt{1+\kappa^2}\exp\left[i\left(\frac{\pi}{2} + \varphi\right)\right]$$

and

$$\triangle + 1 = 1 + ik = \sqrt{1+\kappa^2}\exp\left[i\left(\frac{\pi}{2} - \varphi\right)\right]\,.$$

Condition (6.15) becomes

$$\frac{\widetilde{c}}{\widetilde{b}} = \left(\exp\left[i\left(\frac{\pi}{2} + \varphi\right) - i\left(\frac{\pi}{2} - \varphi\right)\right]\right)^{2/i\kappa} = e^{\frac{4\varphi}{\kappa}}\,. \tag{6.17}$$

As g increases from $\nu^2/8$ to ∞, κ increases from 0 to ∞, and the ratio $\widetilde{c}/\widetilde{b}$ decreases from ∞ to 1.

Let us now make the additional assumption that Π is a symmetric function,

$$\Pi(s_1, s_2) = \Pi(s_2, s_1)\,. \tag{6.18}$$

Then we must have

$$\widetilde{b}\widetilde{c} = 1\,. \tag{6.19}$$

Thus, in the first case, it follows from (6.9) that

$$\widetilde{c} = \frac{1}{\widetilde{b}} = \left(\frac{\theta}{\theta-1}\right)^{1/(2\theta-1)} = \left(\frac{\triangle+1}{\triangle-1}\right)^{1/\triangle}, \tag{6.20}$$

and in the second case, from (6.17) that

$$\widetilde{c} = \frac{1}{\widetilde{b}} = e^{\frac{2\varphi}{\kappa}}. \tag{6.21}$$

We note that $\widetilde{b}$ and $\widetilde{c}$ in (6.21) satisfy condition (3.22).

Remark. The family of homogeneous payoff functions of degree one satisfying (6.2) and (6.3) can be described explicitly. From these conditions and (4.1), it follows that

$$\pi'(b) = 0 \qquad \text{for} \quad 0 < b < 1$$

and

$$\pi(c) - c\pi'(c) = 0 \qquad \text{for} \quad 1 < c < \infty\,.$$

Thus $\pi(x) = K_1$ for $0 < x < 1$ and $\pi(x) = K_2 x$ for $1 < x < \infty$, where K_1 and K_2 are two constants, or

$$\Pi(s_1, s_2) = \begin{cases} K_1 s_2 & \text{for } s_1 < s_2 \\ K_2 s_1 & \text{for } s_1 > s_2\,. \end{cases} \tag{6.22}$$

Therefore, the additional assumption of symmetry is equivalent to assuming $K_1 = K_2$, i.e., that the payoff function is a multiple of (6.1). Similarly, symmetric homogeneous payoff functions satisfying (6.11) and (6.12) must be multiples of (6.10).

7 Symmetric Payoff Functions and Reciprocal Exercise Ratios

In this section we assume that the function Π is symmetric, i.e., (6.18) holds, or equivalently,

$$\pi(x) = x\pi(1/x)\,, \qquad x > 0\,. \tag{7.1}$$

Then, for $0 < b < 1$,

$$V(S_1, S_2; b, 1/b) = V(S_2, S_1; b, 1/b)\,, \tag{7.2}$$

and (6.6) simplifies to

$$V(S_1, S_2; b, 1/b) = w(b)(S_1^\theta S_2^{1-\theta} + S_1^{1-\theta} S_2^\theta). \tag{7.3}$$

From the boundary condition (5.10) we obtain

$$w(b) = \frac{\pi(b)}{b^\theta + b^{1-\theta}}. \tag{7.4}$$

Thus, for $0 < b < 1$,

$$V(S_1, S_2; b, 1/b) = \pi(b) \frac{S_1^\theta S_2^{1-\theta} + S_1^{1-\theta} S_2^\theta}{b^\theta + b^{1-\theta}} \tag{7.5}$$

$$= \pi(b) \sqrt{\frac{S_1 S_2}{b}} \frac{\cosh[\triangle \ln(S_1/S_2)/2]}{\cosh[\triangle \ln(b)/2]}. \tag{7.6}$$

The optimal value of $b = \widetilde{b}$ is given by the first order condition

$$w'(\widetilde{b}) = 0, \qquad 0 < \widetilde{b} < 1. \tag{7.7}$$

If $\triangle$ is an imaginary number, $\triangle = i\kappa$, $\kappa > 0$, then (7.6) can be written as

$$V(S_1, S_2; b, 1/b) = \pi(b) \sqrt{\frac{S_1 S_2}{b}} \frac{\cos[\kappa \ln(S_1/S_2)/2]}{\cos[\kappa \ln(b)/2]}. \tag{7.8}$$

Similarly, it follows from condition (5.11) that, for $c > 1$,

$$V(S_1, S_2; 1/c, c) = \pi(c) \frac{S_1^\theta S_2^{1-\theta} + S_1^{1-\theta} S_2^\theta}{c^\theta + c^{1-\theta}}. \tag{7.9}$$

Formula (7.1) shows that formulas (7.5) and (7.9) are equivalent.
Consider the special case where $\pi(b)$ is constant, $0 < b < 1$, and $g < 0$; then the denominator in (7.5) is minimal if

$$\theta \widetilde{b}^{\theta-1} + (1-\theta)\widetilde{b}^{-\theta} = 0.$$

Hence the optimal value of b is

$$\widetilde{b} = \left(\frac{\theta - 1}{\theta}\right)^{1/(2\theta-1)}, \tag{7.10}$$

which is the same as (6.20). This is not surprising: if $\pi(b)$ is constant, (6.2) and (6.3) are satisfied. Similarly, if $\pi(c)$ is constant for $c > 1$ and $g > \nu^2/8$, we gather from (7.9) that

$$\widetilde{c} = \left(\frac{\theta - 1}{\theta}\right)^{1/(2\theta-1)}, \tag{7.11}$$

which confirms (6.21).

We conclude this section with the *symmetric Margrabe option*, for which the payoff function is

$$\Pi(s_1, s_2) = |s_1 - s_2|, \tag{7.12}$$

so that

$$\pi(b) = 1 - b, \qquad 0 < b < 1. \tag{7.13}$$

We assume that $g < 0$. Hence we maximize

$$w(b) = (1-b)/(b^{\theta} + b^{1-\theta}), \qquad 0 < b < 1. \tag{7.14}$$

There is no explicit formula for the optimal option-exercise ratio $\widetilde{b}$, which has to be determined numerically. The following table allows a quick pricing of the symmetric Margrabe option:

θ_1	$\widetilde{b}$	$w(\widetilde{b})$
1.1	.058	.686
1.2	.118	.548
1.3	.173	.461
1.4	.223	.400
1.5	.268	.354
1.6	.309	.318
1.7	.345	.289
1.8	.378	.265
1.9	.408	.244
2.0	.435	.227
2.1	.460	.212
2.2	.483	.199
2.3	.504	.188
2.4	.524	.177
2.5	.542	.168
2.6	.559	.160
2.7	.574	.153
2.8	.589	.146
2.9	.602	.140
3.0	.615	.134
4.0	.708	.0951
5.0	.765	.0739

(continuation)

θ_1	$\widetilde{b}$	$w(\widetilde{b})$
6.0	.804	.0604
7.0	.831	.0511
8.0	.852	.0442
9.0	.868	.0390
10.0	.881	.0349
11.0	.892	.0316
12.0	.901	.0288
13.0	.908	.0265

Remark. In the special case where $S_2(t) = Ke^{\delta t}$, the symmetric Margrabe option can be interpreted as an indexed *perpetual straddle* (on stock 1), with an exercise price that is compounded with interest: if the option is exercised at time t, a payment of amount

$$e^{gt}|S_1(t) - Ke^{\delta t}|$$

is made at that time.

8 Pricing the Indexed Maximum and Minimum Options

We are now ready to give an explicit formula for the current price of a contract for a single payment of amount

$$P(t, S_1(t), S_2(t)) = e^{gt}\,\mathrm{Max}(S_1(t), S_2(t))\,, \tag{8.1}$$

if it is paid a time $t, t \geq 0$. Here, $g < 0$ and the timing of the payment is controlled by the payee (creditor). Obviously,

$$\pi(b) = 1\,, \qquad 0 < b < 1\,.$$

The optimal contract-exercise ratio $\widetilde{b}$ is given by (6.20) or (7.10). For $\widetilde{b} \leq S_1/S_2 \leq 1/\widetilde{b}$, it follows from (7.5) that the current price is

$$V(S_1, S_2; \widetilde{b}, 1/\widetilde{b}) = \frac{S_1^{\theta}S_2^{1-\theta} + S_1^{1-\theta}S_2^{\theta}}{\widetilde{b}^{\theta} + \widetilde{b}^{1-\theta}} \tag{8.2}$$

$$= \left(\frac{\Delta+1}{\Delta-1}\right)^{1/2\Delta} \frac{S_1^{\theta}S_2^{1-\theta} + S_1^{1-\theta}S_2^{\theta}}{\sqrt{\frac{\Delta-1}{\Delta+1}} + \sqrt{\frac{\Delta+1}{\Delta-1}}}\,. \tag{8.3}$$

Next, we consider the *dual* problem. We want to price a financial contract for a single payment of amount

$$P(t, S_1(t), S_2(t)) = e^{gt}\,\mathrm{Min}(S_1(t), S_2(t))\,, \tag{8.4}$$

if it is paid at time $t, t \geq 0$. Here, $g > \nu^2/8$ and the timing of the payment is controlled by the payer (debitor). As compared with the *primal* problem, the role of b is switched with that of c. The optimal contract-exercise ratio $\widetilde{c}$ is given by (6.21). For $1/\widetilde{c} \leq S_1/S_2 \leq \widetilde{c}$, the current price of the contract is

$$\begin{aligned} V(S_1, S_2; 1/\widetilde{c}, \widetilde{c}) &= \frac{S_1^\theta S_2^{1-\theta} + S_1^{1-\theta} S_2^\theta}{\widetilde{c}^\theta + \widetilde{c}^{1-\theta}} \\ &= \sqrt{\frac{S_1 S_2}{\widetilde{c}}}\,\frac{\cos[\kappa \ln(S_1/S_2)/2]}{\cos[\kappa \ln(\widetilde{c})/2]}\,. \end{aligned} \tag{8.5}$$

Applying (6.21) yields

$$\cos[\kappa \ln(\widetilde{c})/2] = \cos(\varphi) = \frac{1}{\sqrt{1+\kappa^{-2}}}\,,$$

and

$$V(S_1, S_2; 1/\widetilde{c}, \widetilde{c}) = e^{-\varphi/\kappa}\sqrt{(1+\kappa^{-2})S_1 S_2}\,\cos[\kappa \ln(S_1/S_2)/2]\,. \tag{8.6}$$

We note that this value tends to 0 as g decreases to $\nu^2/8$ (κ decreases to 0, φ increases to $\pi/2$). This shows that, for $g \leq \nu^2/8$, the infimum over all stopping times T of

$$E[e^{-(\delta-g)T}\,\mathrm{Min}(S_1(T), S_2(T))]$$

is zero.

Remark. Appropriate indexing is important. For an illustration, consider the indexed maximum option. As g increases to 0 ($\triangle$ decreases to 1), the option price, given by (8.2) and (8.3), tends to $S_1 + S_2$. But then $\widetilde{b}$, given by (6.20), tends to 0, which shows that no optimal option-exercise strategy exists in the limiting case.

9 Pricing the Indexed Russian Option

In this section we consider a single non-dividend-paying stock. Let $S(t)$ denote its price at time $t, t \geq 0$. We assume that $\{\ln[S(t)/S(0)]\}$ is a Wiener process with

$$\mathrm{Var}[\ln S(t)] = \sigma^2 t\,, \tag{9.1}$$

and that $\{e^{-\delta t}S(t); t \geq 0\}$ is a martingale. We also write $S = S(0)$. Let M be a number such that $M \geq S = S(0)$. Define

$$M(t) = \mathrm{Max}\{Me^{\delta t}, \mathrm{Max}[S(u)e^{\delta(t-u)} \mid 0 \leq u \leq t]\}\,, \tag{9.2}$$

Note that the pair $\{S(t), M(t); t \geq 0\}$ is a time-homogeneous Markov process. For $g < 0$, we consider a financial contract for a single payment of amount

$$e^{gt}M(t)\,,$$

if it is paid at time $t, t \geq 0$. Note that this amount is path dependent! The timing of the payment is controlled by the payee (creditor). Thus the current price of the contract is the supremum, taken over all stopping times T, of the expectation

$$E[e^{-(\delta-g)T}M(T)]\,. \tag{9.3}$$

We consider contract-exercise strategies of the form

$$T_b = \mathrm{Min}\{t \mid S(t) = bM(t)\}$$

with $0 < b < 1$. The value of such a strategy is

$$V(S, M; b) = E[e^{-(\delta-g)T_b}M(T_b)]\,,$$

for $bM \leq S \leq M$. Note that the function $V(S, M; b)$ is homogeneous of degree 1 in the variables S and M. We shall derive a functional equation for V. For this purpose we suppose that $bM < S < M$, and we distinguish according to whether the stock price $S(t)$ first reaches $bM(t)$ or $M(t)$. With the identification

$$\begin{array}{llll} S_1(t) = S(t), & S_2(t) = Me^{\delta t}, & \nu = \sigma, & c = 1 \\ \pi(b) = 1, & \pi(c) \; = V(1, 1; b)\,, & & \end{array}$$

it follows from (4.4) that

$$V(S, M; b) = \beta(S, M; b, 1) + V(1, 1; b)\gamma(S, M; b, 1)\,. \tag{9.4}$$

Now we substitute (3.16) and (3.17) and get

$$V(S,M;b) = \frac{1}{b^{1-\theta} - b^{\theta}}\{[S^{1-\theta}M^{\theta} - S^{\theta}M^{1-\theta}] + V(1,1;b)[S^{\theta}(bM)^{1-\theta} - S^{1-\theta}(bM)^{\theta}]\}\,. \tag{9.5}$$

The value of $V(1,1;b)$ may be determined by an appropriate boundary condition at $S = M$. Such a condition can be derived by the following heuristic argument. If the current stock price S is very close to M, we can be "almost sure" that $S(t)$ will exceed $Me^{\delta t}$ before it is optimal for the payee to demand payment. In this sense $M(T_b)$ does not "depend" on the precise value of M, and we conclude that

$$V_M(M,M;b) = 0\,. \tag{9.6}$$

See also Theorem 3 of Goldman, Sosin and Gatto (1979).

It follows from (9.5) and (9.6) that

$$[\theta - (1-\theta)] + V(1,1;b)[(1-\theta)b^{1-\theta} - \theta b^{\theta}] = 0\,,$$

or

$$V(1,1;b) = \frac{1-2\theta}{(1-\theta)b^{1-\theta} - \theta b^{\theta}}\,. \tag{9.7}$$

Substituting (9.7) into (9.5) and simplifying yields

$$V(S,M;b) = \frac{(1-\theta)S^{1-\theta}M^{\theta} - \theta S^{\theta}M^{1-\theta}}{(1-\theta)b^{1-\theta} - \theta b^{\theta}}\,. \tag{9.8}$$

To maximize this expression (as a function of b), we minimize the denominator, whose derivative is

$$(1-\theta)^2 b^{-\theta} - \theta^2 b^{\theta-1}\,.$$

Thus the optimal value of b is

$$\widetilde{b} = \left(\frac{\theta-1}{\theta}\right)^{2/(2\theta-1)} = \left(\frac{\triangle-1}{\triangle+1}\right)^{2/\triangle}\,. \tag{9.9}$$

Note the striking analogy between (9.9) and (6.9).

10 Pricing the Dual Indexed Russian Option

Here we study the dual problem of the one discussed in the last section. For $0 < m \leq S$, define

$$m(t) = \mathrm{Min}\{me^{\delta t}, \mathrm{Min}[S(u)e^{\delta(t-u)} \mid 0 \leq u \leq t]\}. \tag{10.1}$$

For $g > 0$, we consider a financial contract for a single payment of amount

$$e^{gt}m(t),$$

if it is paid at time t, $t \geq 0$. The timing of the payment is controlled by the payer (debitor). Thus the current price of the contract is the infimum, taken over all stopping times T, of the expectation

$$E[e^{-(\delta-g)T}m(T)]. \tag{10.2}$$

We consider contract-exercise strategies of the form

$$T_c = \mathrm{Min}\{t \mid S(t) = cm(t)\},$$

with $c > 1$. For $m < S < cm$, the value of such a strategy is

$$\begin{aligned} V(S, m; c) &= E[e^{-(\delta-g)T_c}m(T_c)] \\ &= V(1, 1; c)\beta(S, m; 1, c) + \gamma(S, m; 1, c) \\ &= \frac{1}{c^\theta - c^{1-\theta}}\{V(1, 1; c)[S^{1-\theta}(cm)^\theta - S^\theta(cm)^{1-\theta}] \\ &\qquad + [S^\theta m^{1-\theta} - S^{1-\theta}m^\theta]\}. \end{aligned}$$

From the condition

$$V_m(m, m; c) = 0, \tag{10.3}$$

we obtain

$$V(1, 1; c)[\theta c^\theta - (1-\theta)c^{1-\theta}] + [(1-\theta) - \theta] = 0,$$

or

$$V(1, 1; c) = \frac{1 - 2\theta}{(1-\theta)c^{1-\theta} - \theta c^\theta}. \tag{10.4}$$

Upon substitution and simplification we get the formula

$$V(S, m; c) = \frac{(1-\theta)S^{1-\theta}m^\theta - \theta S^\theta m^{1-\theta}}{(1-\theta)c^{1-\theta} - \theta c^\theta}, \tag{10.5}$$

valid for $c > 1$, with the additional restriction (3.22) (with $b = 1$) when $\triangle$ is imaginary, and $m \leq S \leq cm$.

So far, the development is parallel to that of the last section. But now the discussion of the denominator is more delicate. If $0 < g < \sigma^2/8$, then

$$0 < \theta_0 < \frac{1}{2} < \theta_1 < 1 .$$

By considering $\theta = \theta_0$ in (10.5), we see that the derivative (with respect to c) of the denominator is positive. Hence $V(S, m; c)$ is the smaller, the larger c is, and an optimal value of c does not exist.

If $g > \sigma^2/8$, then $\theta_0 = 1/2 - i\kappa/2$ and $\theta_1 = 1/2 + i\kappa/2$, $\kappa > 0$. The first order condition

$$V_c(S, m; \widetilde{c}) = 0 \tag{10.6}$$

leads to

$$\widetilde{c} = e^{4\varphi/\kappa} , \tag{10.7}$$

where $\varphi = \operatorname{arccot}(\kappa)$, $0 < \varphi < \pi/2$, as defined in (6.16).

For further discussion, we write (10.5) as

$$V(S, m; c) = \sqrt{\frac{Sm}{c}} \frac{\sin([\kappa \ln(S/m)/2] - \varphi)}{\sin([\kappa \ln(c)/2] - \varphi)} . \tag{10.8}$$

Apply (10.7) to (10.8); then for $S/m \leq e^{4\varphi/\kappa}$, the current price of the contract is

$$\begin{aligned} V(S, m; \widetilde{c}) &= \frac{\sqrt{Sm}}{e^{2\varphi/\kappa}} \frac{\sin([\kappa \ln(S/m)/2] - \varphi)}{\sin(\varphi)} \\ &= e^{-2\varphi/\kappa} \sqrt{(1 + \kappa^2) Sm} \, \sin([\kappa \ln(S/m)/2] - \varphi) . \end{aligned} \tag{10.9}$$

As g decreases to $\sigma^2/8$ (κ decreases to 0, φ increases to $\pi/2$), $V(S, m; \widetilde{c})$ tends to 0; hence we conclude that, if $g \leq \sigma^2/8$, the infimum of the expectation (10.2) is zero.

11 An Alternative Derivation of the Indexed Russian Option Price

The analogy between (6.9) and (9.9) motivates the following alternative derivation of the price of the indexed Russian option. It follows from (9.4) [and (6.6)] that

$$V(S, M; b) = w_1(b, 1) S^\theta M^{1-\theta} + w_2(b, 1) S^{1-\theta} M^\theta , \tag{11.1}$$

where the coefficients w_1 and w_2 do not depend on S or M. The first order condition

$$V_b(S, M; \tilde{b}) = 0 \tag{11.2}$$

is equivalent to the smooth pasting condition

$$V_S(\tilde{b}M, M; \tilde{b}) = 0, \tag{11.3}$$

or

$$w_1(\tilde{b}, 1)\theta(1/\tilde{b})^{1-\theta} + w_2(\tilde{b}, 1)(1-\theta)(1/\tilde{b})^{\theta} = 0 \tag{11.4}$$

[which is analogous to (6.7)]. The boundary condition (9.6), with $b = \tilde{b}$, yields

$$w_1(\tilde{b}, 1)(1-\theta) + w_2(\tilde{b}, 1)\theta = 0 \tag{11.5}$$

[which is analogous to (6.8)]. Formula (9.9) can now be obtained from (11.4) and (11.5).
Since

$$V(bM, M; b) = M,$$

we have

$$w_1(b, 1)b^{\theta} + w_2(b, 1)b^{1-\theta} = 1. \tag{11.6}$$

Using (11.5) and (11.6) (with $b = \tilde{b}$) we obtain formula (9.8) with $b = \tilde{b}$. On the other hand, using (11.4) and (11.6) we obtain

$$V(S, M; \tilde{b}) = \frac{1}{2\theta - 1}[(\theta - 1)(S/\tilde{b})^{\theta}M^{1-\theta} + \theta(S/\tilde{b})^{1-\theta}M^{\theta}], \tag{11.7}$$

which is analogous to (2.4) of Shepp and Shiryaev (1993a).

12 Concluding Remarks

In this paper we study the pricing of financial contracts for a single indexed payment, which is a homogeneous function (of degree one) of two stock prices. Our approach is to take advantage of the stationary nature of the problem, construct two martingales (with respect to the risk-neutral measure), and apply the optional sampling theorem.

We have proposed an alternative to the American-type options with a finite expiration date. By indexing the payoff appropriately, we can make sure that the option will be exercised, even if it is a perpetual option on non-dividend-paying stocks. We have shown how such an option can be analyzed in a transparent fashion.

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Summary

We consider a contract for a payment that is an indexed function of the prices of two stocks or assets. The duration of the contract is unlimited. We postulate an exponential decay of the index, if the creditor can choose the time of the payment, and an exponential growth of the index, if the debitor can choose it. The price of such a contract can be obtained in a transparent fashion, if the payoff function is a homogeneous function (of degree one) of the stock prices: then the optimal exercise strategies depend only on the ratios of the stock prices. Explicit results are obtained for the options on the indexed minimum and maximum of the two stock prices and for the symmetric Margrabe option, where the payment is the indexed absolute value of their difference. We also examine the pricing of the indexed Russian option, where the payment is the observed indexed maximum of a single stock, and its dual, where the payment is related to the observed indexed minimum.
By applying martingale methods, we avoid differential equations. We also derive the smooth pasting conditions, which determine the optimal contract-exercise ratios of the stock prices.

Zusammenfassung

Wir betrachten einen zeitlich unbeschränkten Vertrag für eine Zahlung, deren Betrag eine mit einem Index multiplizierte Funktion der Kurse zweier Aktien ist. Falls der Schuldner den Zeitpunkt der Zahlung wählen kann, setzt man ein exponentielles Wachstum des Index voraus; falls der Gläubiger diesen Zeitpunkt bestimmen kann, nimmt man an, dass der Index exponentiell abfällt. Der Preis eines solchen Vertrages kann auf transparente Art bestimmt werden, falls die Auszahlungsfunktion eine homogene Funktion der beiden Aktienkurse ist: dann hängt die optimale Ausübungsstrategie der Option lediglich vom Verhältnis der beiden Aktienkurse ab. Explizite Resultate werden in Spezialfällen erhalten: falls der Betrag der Zahlung das indizierte Minimum oder Maximum ist, und für die symmetrische Margrabe-Option, bei welcher der Betrag der indizierte Absolutbetrag der Differenz beträgt. Ferner analysieren wir den Preis der indizierten Russischen Option (welche auf dem historischen Höchstkurs einer Aktie beruht) und die entsprechende duale Option.
Dank Martingalemethoden können Differentialgleichungen vermieden werden. Ferner leiten wir die sogenannten "smooth pasting conditions" (die stetigen Verheftungsbedingungen) her, anhand von welchen die für die Ausübung der Option optimalen Quotienten der Aktienkurse bestimmt werden können.

Résumé

L'article considère un contrat de durée illimitée comportant un paiement dont le montant est une fonction des cours de deux actions, multipliée par un indice. Si c'est le débiteur qui peut choisir le moment du paiement, le modèle suppose une croissance exponentielle de l'indice; si par contre c'est le créancier qui peut choisir le moment du paiement, le modèle fait intervenir une décroissance exponentielle de l'indice. Le prix d'un tel contrat peut être déterminé sans autre lorsque la fonction intervenant dans la détermination du montant du paiement est une fonction homogène des deux cours des actions considérées: dans ce cas la stratégie d'exercice optimale de l'option dépend simplement du rapport des cours des deux actions. Des résultats explicites peuvent être obtenus dans des cas particuliers: lorsque le montant du paiement est le minimum ou le maximum indexé, et, pour l'option

Margrabe symétrique, lorsque le montant du paiement est égal à la valeur absolue indexée de la différence des deux cours. Par ailleurs les auteurs analysent le prix de l'option russe indexée (qui se base sur le cours historique maximum de l'action) et de l'option duale correspondante.
Graĉe à des méthodes utilisées en théorie des martingales, il est possible d'éviter des équations différentielles. Enfin les auteurs établissent les conditions pour les raccordements continus (connues sous le nom de "smooth pasting conditions"), conditions qui permettent de déterminer les quotients optimaux des cours des actions.