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Splitting Risk and Premium Calculation

Introduction

The effect on premium calculation of splitting a risk into two components is analyzed. A brief outline of the study follows.

Measuring the total splitting risk using the variance, it is shown in Section 1 that the maximum variance premium reduction is equal to half of the loading. This optimal variance reduction is attained for a linear risk-exchange, in which the mean level of the retained risk can be chosen and half of the realized mean claims deviation is exchanged. It is also possible to attain a maximum variance reduction through excess-of-loss or stop-loss reinsurance, but only under well-defined extremal distributions, for example under the so-called distribution of Bowers.

In Section 2 the total splitting risk is measured using the standard deviation. The corresponding premium loading can never be decreased through splitting. The most favorable status quo is reached for a linear risk-exchange or for a stop-loss reinsurance under Bowers' distribution.

From Section 3 on we are interested in the design of a premium calculation principle, which is additive for independent risks and for splitting risk components. If arbitrage opportunities should be avoided in a risk-exchange economy, then these latter properties must hold. It is shown in a special case that a premium principle with these additive properties, and characterized by the values it takes on the set of diatomic risks with given mean and variance, necessarily satisfies the CAPM (= **C**apital **A**ssert **P**ricing **M**odel) like relationships first proposed by Borch (1982). The rest of the paper is devoted to the analysis of some consequences for premium calculation this premium rating device does have.

In Section 4 a set of feasible reinsurance contracts with a fixed maximum deductible is considered. It is shown that such a contract induces an experience rated insurance contract, which offers a well-defined perfectly matched dividend or bonus. The associated "fair premium" equals the sum of the expected claims, the expected amount of dividend payments and the loading for reinsurance. Since the guaranteed dividend belongs to the insured, one can consider a "mean risk premium" obtained by subtracting the expected dividend from the fair premium, and which correspond to a risk premium needed in the average. The loading of the mean risk premium equals the reinsurance loading. If the price of reinsurance is known, "optimal" dividends can be found. If the price of reinsurance is not known, one uses the CAPM relationships of Section 3 and the fact that the net outcome of the direct insurer is immunized (perfect hedge) to derive parameter-free CAPM based formulas for the fair premium, the reinsurance premium and the mean risk premium. In the special case of stop-loss reinsurance and non-negative risks, the limiting CAPM based fair premium as the dividend payment goes to zero identifies with the distribution-free and parameter-free modified variance premium first advocated by Heilmann (1988), which finds herewith an insurance economics interpretation. In the general stop-loss case lower and upper bounds for the CAPM based premiums are derived.

Finally in Section 5 the CAPM based reinsurance premium is replaced by a safe diatomic estimate and the behaviour of the associated mean risk premium is studied for the typical case of a stop-loss reinsurance and non-negative risks. Interesting distribution-free results, which depend only on the mean and the coefficient of variation of the risk, are obtained.

1. The total splitting risk a measured by variance

The following question is of practical interest. Given a risk X with associated risk premium $P = H[X]$, is it possible to split up the risk in smaller parts X_i with premiums $P_i = H[X_i]$, $i = 1, \ldots, n$, such that $X_1 + \cdots + X_n = X$ and $P_1 + \cdots + P_n < P$? In case this is possible, what is the maximum possible premium reduction? The case $n = 2$, which represents a risk-exchange or reinsurance between two risk takers, is somewhat analyzed. A forrunner is Hürlimann (1994a), which considers solely the special of excess-of-loss or stop-loss reinsurance.

Let $Y = u(X), Z = v(X)$ be transformations of the random variable X such that $Y + Z = X$ and assume premiums are set according to the variance calculation principle with loading factor θ, that is $H[\cdot] = E[\cdot] + \theta\sigma^2[\cdot]$, where $E[\cdot]$, $\sigma^2[\cdot]$ are the functionals which take expected values and variances. Without splitting the risk premium is thus $P = H[X] = E[X] + \theta\sigma^2[X]$ and with splitting it is $Q = H[Y] + H[Z] = E[X] + \theta R[Y, Z]$, where $R[Y, Z] = \sigma^2[Y] + \sigma^2[Z]$ is called the *total splitting risk* (as measured by variance) of the insurance risk X.

In this situation the maximum premium reduction is given by the following elementary result.

Proposition 1.1. Let $Y = u(X)$ and $Z = v(X)$ be transformed random variables of X such that $Y + Z = X$. Then the total (variance) splitting risk satisfies the following best lower bound

$$R[Y,Z] \geq \frac{1}{2}\sigma^2[X]\,, \tag{1.1}$$

which is attained by the linear transformation

$$\begin{aligned} Y &= \frac{1}{2}(X - E[X]) + E[Y] \\ Z &= \frac{1}{2}(X - E[X]) + E[Z] \end{aligned} \tag{1.2}$$

Proof. Since $R[Y,Z] = \sigma^2[X] - 2\,\mathrm{Cov}[Y,Z]$, to get a minimum, it suffices to maximize the covariance. Applying probabilistic approximation results, one can directly solve this optimization problem for a discrete random variable X concentrated on finitely many atoms. However this procedure involves some analytical calculations. The following simpler argument is due to the referee. By Cauchy-Schwarz one has the inequality

$$\mathrm{Cov}[Y,Z] \leq \sqrt{\sigma^2[Y] \cdot \sigma^2[Z]}\,,$$

which is attained if and only if $Y = aZ + b$ almost surely. Since $X = Y + Z$ this choice yields the relation

$$Z = \frac{X - b}{a + 1}\,.$$

It follows that

$$R[Y,Z] = \frac{a^2 + 1}{(a+1)^2}\sigma^2[X]\,,$$

which is minimal for $a = 1$. Taking expected values in the relation $(a + 1)Z = X - b$ determines the constant b, and the assertion follows.

Remarks 1.1.

(i) The simple inequality (1.1) do not depend on the distribution of X, that is on the underlying probabilistic measure.

(ii) In terms of insurance market theory, the competitive variance risk premium needs actually half the loading of the usual full variance premium in case splitting in only two parts is allowed.

(iii) In a distribution-free framework, the maximum guaranteed variance reduction is realized for a linear risk exchange of the form (1.2). To achieve the optimal variance reduction, an insurance company can fix the mean level $E[Y]$ of the retained business and exchange with a partner half of the realized mean claims deviation.

(iv) Clearly it is possible to iterate this simple splitting scheme. After n iteration steps, a given risk X is split into 2^n components and the corresponding total splitting risk may be reduced at most to $2^{-n}\sigma^2[X]$. Choosing n sufficiently large, the total (variance) splitting risk can be made arbitrarily small. The effect of splitting risks into 2^n parts on the variance premium is thus similar to the effect of merging 2^n independent and identically distributed risks on the standard deviation premium.

As next question it is natural to ask if the above optimal variance reduction can be achieved through reinsurance, which is a restricted form of risk-exchange. In reinsurance theory one usually restricts the set of transformations $u(x), v(x)$ to those compensation functions where neither the cedant nor the reinsurer will benefit in case the claim amount increases. For positive risks X this means that feasible reinsurance contracts can be described by the class of transformed random variables

$$\begin{aligned}\mathrm{Com}(X) = \{(Y, Z) : Y = u(X), Z = v(X) \\ &\text{are comonotonic random variables such that} \\ &Y + Z = X\}\,.\end{aligned}$$

In other words $u(x), v(x)$ are non-decreasing functions such that $0 \le u(x), v(x) \le x$ and $u(x) + v(x) = x$. The relevance of a general notion of comonotonicity in insurance premium calculation theory, which stems from non-expected utility theory, has been recognized by Denneberg (1985/90).

Proposition 1.2. Let X be a random variable taking non-negative values and let $(Y, Z) \varepsilon \,\mathrm{Com}(X)$ be a feasible reinsurance contract. Then the total (variance) splitting risk satisfies the following best lower and upper bounds

$$\frac{1}{2}\sigma^2[X] \le R[Y, Z] \le \sigma^2[X]\,. \tag{1.3}$$

Moreover the lower bound is attained by the proportional reinsurance treaty

$$Y = Z = \frac{1}{2}X\,, \tag{1.4}$$

and the upper bound is attained if and only if Y or Z is a degenerate random variable.

Proof. Tchebycheff's inequality says that for $(Y, Z) \in \mathrm{Com}(X)$ one has $\mathrm{Cov}[Y, Z] \geq 0$ with equality sign if and only if Y or Z is a constant (e.g. Hardy, Littlewood and Polya (1934)), no. 43). This proves the affirmation about the best upper bound. For the lower bound, observe that the solution (1.2) satisfies the constraint $0 \leq Y, Z \leq X$ only if (1.4) holds.

The maximum variance premium reduction through reinsurance is attained by a very special proportional reinsurance treaty and this result is again independent of any distributional assumptions.

However in the real-world many other risk-exchange forms and reinsurance contracts appear and their premium rating often relies on distributional properties of the risks. Therefore the above results are only of a limited practical value. For this reason let us turn to the next immediate questions:

(Q1) For a given feasible reinsurance contract $(Y, Z) \in \mathrm{Com}(X)$, find extremal distributions such that the extreme total splitting risk bounds in Proposition 1.2 are attained.

(Q2) For a given distribution $F(x) = \Pr(X \leq x)$ of the risk, find the extreme total splitting risk bounds over a subclass of $\mathrm{Com}(X)$.

The detailed study of these questions goes beyond the scope of this paper. However a partial answer to question (Q1) in case $Z = (X - d)_+$ is an excess-of-loss or stop-loss reinsurance with deductible d has been observed in Hürlimann (1994a).

Proposition 1.3. Let X be a random variable with mean $\mu = E[X]$ and variance $\sigma^2 = \sigma^2[X]$ and let $Z = (X - d)_+$. Then the following statements hold:

(i) For arbitrary $d \in \mathbb{R}$ one has $R[Y, Z] = \frac{1}{2}\sigma^2$ provided the probabilistic measure follows Bowers' distribution

$$F^B(x) = \frac{1}{2}\left(1 + \frac{x - \mu}{\sqrt{\sigma^2 + (x - \mu)^2}}\right), \qquad x \in (-\infty, \infty). \tag{1.5}$$

(ii) For a fixed $d \leq \mu$ one has $R[Y, Z] = \sigma^2$ in case X is a diatomic random variable with support $\{x_1, x_2\}$ and probabilities $\{p_1, p_2\}$ specified by

$$\begin{aligned} x_1 &= d\,, & p_1 &= \frac{\sigma^2}{\sigma^2 + (d-\mu)^2}, \\ x_2 &= \mu + \frac{\sigma^2}{\mu - d^2}, & p_2 &= 1 - p_1\,. \end{aligned} \tag{1.6}$$

Remarks 1.2.

(i) For fixed $d \in \mathbb{R}$ the minimum total splitting risk is also attained by the diatomic random variable

$$\begin{aligned} x_1 &= d - \sqrt{\sigma^2 + (d-\mu)^2}\,, & p_1 &= \frac{1}{2}\left(1 + \frac{d-\mu}{\sqrt{\sigma^2 + (d-\mu)^2}}\right), \\ x_2 &= d + \sqrt{\sigma^2 + (d-\mu)^2}, & p_2 &= 1 - p_1\,. \end{aligned} \tag{1.7}$$

By the way observe that (1.5) and (1.7) maximize the corresponding net stop-loss premiums (inequality of Bowers (1969)). In contrast to the distribution structure (1.7), which depends on d, the distribution of Bowers (1.5) yields a result valid uniformly for all deductibles.

(ii) In case $d > \mu$ a maximizing distribution is not known to the author.

2 The total splitting risk as measured by standard deviation.

It is clear that the premium reduction observed in Section 1 depends on the chosen premium calculation principle. Premium reduction through risk splitting is not always possible. Let us illustrate with the standard deviation principle $H[\cdot] = E[\cdot] + \theta\sigma[\cdot]$. The splitting premium is now $Q = E[X] + \theta S[Y, Z]$ with the total splitting risk, as measured by standard deviation, equal to $S[Y, Z] = \sigma[Y] + \sigma[Z]$. Through squaring one gets

$$S[Y, Z]^2 = \sigma^2[X] + 2(1 - \varrho(Y, Z))\sigma[Y]\sigma[Z]\,. \tag{2.1}$$

where $\varrho(Y, Z)$ is the correlation coefficient between Y and Z. It follows that

$$S[Y, Z] \geq \sigma[X]\,. \tag{2.2}$$

In this case splitting of risk does not lead to premium reduction. The most favorable situation, namely splitting premium equal to standard deviation premium, occurs when $\varrho(Y,Z)=1$, which implies a linear transformation $Z=aY+b$ for some $a>0$, $b\in\mathbb{R}$. The following result has been shown.

Proposition 2.1. Let $Y=u(X)$ and $Z=v(X)$ be transformed random variables of X such that $Y+Z=X$. Then the total (standard deviation) splitting risk satisfies the following best lower bound

$$S[Y,Z]\geq\sigma[X], \tag{2.1}$$

which is attained by the linear transformation

$$\begin{aligned} Y &= r(X-E[X])+E[Y] \\ Z &= (1-r)(X-E[X])+E[Z], \qquad r\in(0,1). \end{aligned} \tag{2.2}$$

The next result concerns the restriction to reinsurance contracts and its proof is immediate.

Proposition 2.2. Let X be a random variable taking non-negative values and let $(Y,Z)\in\mathrm{Com}(X)$ be a feasible reinsurance contract. Then the total (standard deviation) splitting risk satisfies the following best lower bound

$$S[Y,Z]\geq\sigma[X], \tag{2.3}$$

which is attained by the proportional reinsurance treaties

$$Y=rX,\quad Z=(1-r)X,\qquad r\in(0,1). \tag{2.4}$$

Let us also mention the following result similar to Proposition 1.3.

Proposition 2.3. Let X be a random variable with mean $\mu=E[X]$ and variance $\sigma^2=\sigma^2[X]$ and let $Z=(X-d)_+$. Then for arbitrary $d\in\mathbb{R}$ one has $S[Y,Z]=\sigma$ provided the probabilistic measure follows Bowers' distribution (1.5).

Proof. Set $\sigma^2(d):=\sigma^2[Z], \tau^2(d):=\sigma^2[Y], \pi(d):=E[Z], \chi(d):=d-\mu+\pi(d)$. From the identity

$$\sigma^2(d)+\tau^2(d)=\sigma^2-2\pi(d)\chi(d),$$

one sees that

$$S[Y,Z]^2=\sigma^2+2(\sigma(d)\tau(d)-\pi(d)\chi(d)).$$

But for Bowers' distribution one has $\pi(d)\chi(d)=\frac{1}{4}\sigma^2$ and $\sigma(d)=\tau(d)=\frac{1}{2}\sigma$, which shows the result.

3 Diatomic approximations to splitting risk premiums

Assume the risk X is split into two transformed components $Y = u(X)$, $Z = v(X)$ such that $Y + Z = X$. In order to avoid arbitrage opportunities, the problem is to design a premium principle $H[\cdot]$ which satisfies the additive property

$$H[X] = H[Y] + H[Z]. \tag{3.1}$$

Indeed suppose on the contrary that for example $H[X] > H[Y]+H[Z]$. Then an insurance market participant could choose to insure X and reinsurer the splitting components Y and Z separately. Its asset position equal $H[X]-(H[Y]+H[Z]) > 0$ while its liability is $-X+(Y+Z) = 0$. It follows that this participant has made a riskless profit, which is inconsistent with an economic equilibrium.
Following the findings of several authors, e.g. Goovaerts et al. (1984), let us restrict our attention to premium calculation principles, which are already characterized by the values they take on the set $D_2[a,b] := D_2([a,b];\mu,\sigma)$ of diatomic risks with given mean μ and standard deviation σ defined on the interval $[a,b]$, $a,b\varepsilon\mathbb{R}$. Let $X\varepsilon D_2[a,b]$ has support $\{x_1,x_2\}$ and probabilities $\{p_1,p_2\}$

$$\begin{aligned} &p_1 = \frac{x_2-\mu}{x_2-x_1}, \quad p_2 = \frac{\mu-x_1}{x_2-x_1}, \quad \sigma^2 = (\mu-x_1)(x_2-\mu). \\ &a \le \mu \le b, \quad 0 \le \sigma^2 \le (\mu-a)(b-\mu). \end{aligned} \tag{3.2}$$

Let $\{z_1 = v(x_1), z_2 = v(x_2)\}$ be the support of the transformed random variable $Z = v(X)$.

Lemma 3.1. Let X, Z be the above diatomic risks. Then one has the bivariate moment formula:

$$\begin{aligned} \mu_{1,n-1}[Z] :&= E[(Z-E[Z])(X-\mu)^{n-1}] \\ &= \left(\frac{z_2-z_1}{x_1-x_1}\right)E[X-\mu)^n], \qquad n = 2,3,\ldots \end{aligned} \tag{3.3}$$

Proof. Using the expression (3.2) for the probabilities, one gets the result as follows:

$$\begin{aligned} E[(Z-E[Z])(X-\mu)^{n-1} &= p_1p_2(z_2-z_1)\{(x_2-\mu)^{n-1}-(x_1-\mu)^{n-1}\} \\ &= \left(\frac{z_2-z_1}{x_2-x_1}\right)\{p_2(x_2-\mu)^n + p_1(x_1-\mu)^n\} \\ &= \left(\frac{z_2-z_1}{x_2-x_1}\right)E[(X-\mu)^n]. \end{aligned}$$

To define $H[\cdot]$ on diatomic risks with the splitting property (3.1), let us follow Hürlimann (1993). First of all let us assume $P = H[X]$ is additive for independent risks. After Borch (1982), formula (2), one has necessarily

$$P = H[X] = \sum_{n=1}^{\infty} \alpha_n \kappa_n \,, \tag{3.4}$$

where α_n are constants and κ_n are the cumulants (or semi-invariants). Now write the cumulants as entire rational functions in the central moments $\mu_j, j = 2, 3, \ldots$, say as linear combination

$$\kappa_n = \sum_{j=2}^{n} a_j \mu_j \,, \qquad n = 2, 3, \ldots \,,$$

where the coefficients a_j may depend on the central moments. Define bivariate cumulants setting

$$\kappa_{1,0}[Z] = E[Z], \quad \kappa_{1,n-1}[Z] = \sum_{j=2}^{\infty} a_j \mu_{1,j-1}[Z], \qquad n = 2, 3, \ldots \,,$$

and similarly for Y. After Borch (1982), formula (5), the premium principle

$$H[Z] = \sum_{n=1}^{\infty} \alpha_n \kappa_{1,n-1}[Z] \,, \tag{3.5}$$

and similarly for $H[Y]$, satisfies the splitting property (3.1). Taking into account (3.3), (3.4) and the definition of the bivariate cumulants, one gets

$$\begin{aligned} H[Z] &= \alpha_1 E[Z] + \left(\frac{z_2 - z_1}{x_2 - x_1}\right) \sum_{n=2}^{\infty} \alpha_n \kappa_n \\ &= \alpha_1 E[Z] + \left(\frac{z_2 - z_1}{x_2 - x_1}\right) (P - \alpha_1 E[X]) \\ &= \alpha z_1 + \left(\frac{z_2 - z_1}{x_2 - x_1}\right) (P - \alpha x_1) \,, \end{aligned} \tag{3.6}$$

where one sets $\alpha = \alpha_1$.
A calculation shows that for diatomic risks

$$\frac{\mathrm{Cov}[X, Z]}{\mathrm{Var}[X]} = \frac{z_2 - z_1}{x_2 - x_1} \,. \tag{3.7}$$

This implies that the splitting premium (3.6) identifies with formula (4) in Borch (1982), which follows from the Capital Asset Pricing Model. Our derivation of formula (3.6) provides a proof of the following result. A premium principle of the form (3.5) with the splitting property (3.1), which is additive for independent risks and which should be characterized by the values it takes on diatomic risks $X \in D_2[a,b]$, has splitting premiums necessarily equal to

$$\begin{aligned} H[Y] &= \alpha E[Y] + \frac{\mathrm{Cov}[X,Y]}{\mathrm{Var}[X]}(H[X] - \alpha E[X]), \\ H[Z] &= \alpha E[Z] + \frac{\mathrm{Cov}[X,Z]}{\mathrm{Var}[X]}(H[X] - \alpha E[X]). \end{aligned} \tag{3.8}$$

In the following Sections some consequences for premium calculation based on this splitting scheme will be presented.

4. Splitting schemes and experience rating

Given a risk X suppose the set of feasible reinsurance contracts is described by the set $\mathrm{Com}(X)$ of comonotonic random variables. Recall that $(Y,Z) \in \mathrm{Com}(X)$ if there exist non-decreasing functions $u(x), v(x)$ such that $Y = u(X), Z = v(X)$ and $u(x) + v(x) = x$. In this notation Z describes the reinsurance payment and Y the retained amount. Let us say a feasible reinsurance has *a maximum deductible* d if the following number exists and is finite

$$d = \sup_{x \in \mathbb{R}} \{u(x)\} < \infty. \tag{4.1}$$

Examples 4.1. A stop-loss contract $Z = (X-d)_+$ has (maximum) deductible d. Proportional reinsurance $Z = (1-r)X, 0 < r < 1$, do not admit a maximum deductible since $u(x) = rx$ goes to infinity as x goes to infinity. A combination of proportional and non-proportional reinsurance $Z = (1-r)X + r(X-T)_+$ has a maximum deductible $d = rT$ and a combination of stop-loss in layers $Z = r(X-L)_+ + (1-r)(X-M)_+$, $M > L$, has a maximum deductible $d = rL + (1-r)M$.

The set of feasible reinsurance contracts with maximum deductible d is denoted by

$$\begin{aligned} V_d = \{&(Y = u(X), Z = v(X)) \in \mathrm{Com}(X) \ \text{such that} \\ &d = \sup_{x \in \mathbb{R}} \{u(x)\} < \infty\}. \end{aligned}$$

One sees that for $(Y, Z) \in V_d$ the function

$$d(x) = d - u(x) = d + v(x) - x \tag{4.2}$$

is always non-negative and defines a transformed random variable $D = d(X)$ such that with probability one

$$d + Z = X + D. \tag{4.3}$$

With $x_d = \inf_{u(x)=d}\{x\}$ one has for all $x \geq x_d$

$$d(x) = 0, \quad v(x) = x - d. \tag{4.4}$$

It is possible to interpret D as *a perfectly hedged experience rated dividend.* Following Hürlimann (1994b) an experience rating contract with premium P offers in general besides claims payment X a bonus or dividend $D[X] \geq 0$, which usually is paid out in case the risk profit $P - X$ is positive. In this situation the liability of the insurer is $X + D$. The financial risk of a loss $X + D > P$ may be quite important. Often it is judicious to split the liability in smaller parts, which are covered by different risk takers. Suppose the direct insurer concludes a risk-exchange with a reinsurer such that its retained liability is $Y + D = X - Z + D$. Then the needed premium $P = P[X + D]$ of the experience rating contract is the sum of the net retained premium $P^N = P^N[Y + D]$ plus the reinsurance premium, that is one has $P = P^N + H^R[Z]$, where $H^R[\cdot]$ is the premium principle of the reinsurer. A main problem for the insurer is to find adequate risk-exchange forms $Z = v(X)$ and dividend formulas $D = D[X]$ such that some desirable rate-making properties are fulfilled (see also Hürlimann (1994c)).
As decision criterion suppose the insurer applies the most popular minimum square loss principle widely used in Insurance (credibility theory, Bühlmann (1967)) and Finance (hedging through sequential regression, Föllmer and Schweizer (1988)), which amounts to minimize the expected square difference between assets and liabilities. In our situation one has to minimize the risk quantity

$$R = E[(P^N - Y - D)^2] = \min. \tag{4.5}$$

over the set of random variables (Y, D) such that $(Y, Z) \in \mathrm{Com}(X)$ and $D = D[X] \geq 0$. From the decomposition $R = (P^N - E[Y + D])^2 + \mathrm{Var}[Y + D]$, one sees that necessarily $P^N = E[Y + D]$. It follows that the expression $R = \mathrm{Var}[Y + D] = \mathrm{Var}[Y] + 2\,\mathrm{Cov}[Y, D] + \mathrm{Var}[D]$ is minimum provided

$\operatorname{Cov}[Y, D] = -\operatorname{Var}[D]$ or $\operatorname{Cov}[Y, D] = -\operatorname{Var}[Y]$. In the first case one has $R_{\min} = \operatorname{Var}[Y] - \operatorname{Var}[D] = \operatorname{Var}[Y]\{1 - \varrho(Y, D)^2\}$, where

$$\varrho(Y, D)^2 = \frac{\operatorname{Cov}[Y, D]^2}{\operatorname{Var}[Y] \cdot \operatorname{Var}[D]} = \frac{\operatorname{Var}[D]}{\operatorname{Var}[Y]} \tag{4.6}$$

is the square of the correlation coefficient between Y and D. The second case is similar. In particular the risk of the direct insurer can be completely eliminated (perfect hedge), that is $R_{\min} = 0$, provided there exists Y, D such that $\operatorname{Cov}[Y, D] = -\operatorname{Var}[D] = -\operatorname{Var}[Y]$. Using the above considerations, a set of perfectly hedged experience rated dividends can be characterized mathematically as follows.

Proposition 4.1. Suppose $(Y, Z) \in \operatorname{Com}(X)$ and $D = D[X] \geq 0$ define an experience rating contract. Assume that the set of real values $\{x \in \mathbb{R} : D[X = x] = 0\}$ is non-empty. Then the following conditions are equivalent:

(C1) One has $P^N = d = \sup\limits_{x \in \mathbb{R}}\{u(x)\}$ and $R_{\min} = E[(P^N - Y - D)^2] = 0$

(C2) One has $\operatorname{Cov}[Y, D] = -\operatorname{Var}[D] = -\operatorname{Var}[Y]$

(C3) $(Y, Z) \in V_d$ defines a feasible reinsurance contract with maximum deductible d and perfectly matched dividend $D = d - Y$.

Proof. That (C1) implies (C2) is contained in the above discussion. Let us show that (C2) implies (C3). Under (C2) one has $\varrho(Y, D)^2 = 1$, from which it follows that $D = cY + d$, $c, d \in \mathbb{R}$, almost surely (see e.g. Fisz (1973), Satz 3.6.5, p. 112). Since $\operatorname{Cov}[Y, D] = -\operatorname{Var}[Y]$, one has $c = -1$. Moreover from $D = d - Y \geq 0$ one deduces that $Y \leq d$, hence $\sup_{x \in \mathbb{R}}\{u(x)\}$ exists and is finite. Since $D = 0$ is attained by assumption, one has necessarily $d = \sup_{x \in \mathbb{R}}\{u(x)\}$, which shows the validity of (C3). Finally it is obvious that (C1) follows from (C3).
Knowing the price of reinsurance, that is the premium calculation principle $H^R[\cdot]$ of the reinsurer, the *fair premium* P of a perfectly matched experience rated contract $(Y, Z) \in V_d$ is uniquely given by

$$P = E[X] + E[D] + (H^R[Z] - E[Z]). \tag{4.7}$$

One observes that besides the expected costs for claims and dividend payments only the loading for reinsurance has to be paid in a "fair" experience rated contract of the above type. This corresponds to the "Dutch property" of the Dutch premium principle (see Van Heerwaarden and Kaas (1992)). Moreover accounting for the direct insurer is very simple and does not involve any risk as Table 1 shows.

Table 1: accounting scheme of the direct insurer

income	
premium	$E[X]+E[D]+(H^R[Z]-E[Z])$
reinsurance payment	Z
outcome	
claims payment	X
guaranteed dividend	D
reinsurance premium	$H^R[Z]$
net outcome	$E[X+D-Z]-(X+D-Z)=E[d]-d=0$

In the above situation the guaranteed dividend payment D belongs to the insured. From his point of view, the average risk premium needed to cover the risk X, which we call *mean risk premium*, is equal to

$$\overline{P}=E[X]+(H^R[Z]-E[Z])\,. \tag{4.8}$$

Besides the expected value of the claims payment only the reinsurance loading is needed in the mean as security loading. However the periodic accounting fluctuations of the insured do not vanish. The remaining periodic risk of the insured is related to the fluctuations of the dividend formula and equal to

$$E[Y]-Y=D-E[D]\,. \tag{4.9}$$

If the price of reinsurance is known, "optimal" dividends can be found as illustrated in the following situation.

Example 4.2. Suppose the reinsurer applies the variance principle $H^R[Z]=E[Z]+\theta_R\sigma^2[Z]$ with a known factor loading θ_R. For a stop-loss contract $Z=(X-d)_+$, the corresponding perfectly matched dividend formula is $D=(d-X)_+$. A minimum fair premium

$$P(d)=\mu+\chi(d)+\theta_R\sigma^2(d)=\min\,. \tag{4.10}$$

is obtained immediately. Since equivalently $P(d)=d+\pi(d)+\theta_R\sigma^2(d)$ one has

$$\begin{aligned} P'(d)&=F(d)(1-2\theta_R\pi(d))\,,\\ P''(d)&=f(d)(1-2\theta_R\pi(d))+2\theta_R F(d)\overline{F}(d)\,. \end{aligned} \tag{4.11}$$

From elementary calculus it follows that the "optimal" deductible is solution of the equation

$$2\theta_R \pi(d) = 1\,, \tag{4.12}$$

which occurs also in Hürlimann (1994b).

In general the price of reinsurance will not be known with certainty, and there remains the problem of calculating adequate reinsurance premiums $H^R[Z]$. Suppose the market price P for covering the risk X is calculated according to the variance premium principle, that is

$$P = H[X] = \mu + \theta\sigma^2 \tag{4.13}$$

for some unknown factor loading θ. In the following we argue that the needed level of the security loading can be determined in case the direct insurer offers a perfectly matched experience rated dividend D belonging to a feasible reinsurance contract with maximum deductible d such that $d + Z = X + D$. As motivated in Section 3 and also justified by the CAPM (see Borch (1982), formula (4)), the risk premium components $H[Y]$, $H[Z]$ of the splitting scheme $X = Y + Z$ can be calculated as follows:

$$\begin{aligned} H[Y] &= E[Y] + \frac{\mathrm{Cov}[X,Y]}{\mathrm{Var}[X]}(P - E[X])\,, \\ H[Z] &= E[Z] + \frac{\mathrm{Cov}[X,Z]}{\mathrm{Var}[X]}(P - E[X])\,. \end{aligned} \tag{4.14}$$

In this context one sets $H^R[Z] = H[Z]$. In this insurance economics interpretation, the net outcome of the direct insurer after payment of the guaranteed dividend is equal to

$$(H[Y] - Y) - D = (H[Y] - d) + (d - Y) - D = H[Y] - d\,. \tag{4.15}$$

But as seen from Table 1 one must have $H[Y] = d$. Comparing with (4.14), the unknown security loading can be eliminated, and one gets the needed *CAPM based fair premium*

$$P = H[X] = E[X] + \frac{\mathrm{Var}[X]}{\mathrm{Cov}[X,Y]}E[D]\,. \tag{4.16}$$

The *CAPM based reinsurance premium* is equal to

$$H^R[Z] = H[Z] = E[Z] + \frac{\mathrm{Cov}[X,Z]}{\mathrm{Cov}[X,Y]}E[D]\,, \tag{4.17}$$

and the corresponding *CAPM based mean risk premium* is given by

$$\overline{P} = E[X] + \frac{\mathrm{Cov}[X,Z]}{\mathrm{Cov}[X,Y]} E[D] . \tag{4.18}$$

An interesting problem consists to determine the *limiting CAPM based fair premium* as the reinsurance payment Z traverses a family of feasible reinsurance contracts with maximum deductible d such that d goes to zero. In the important case of positive risks $X \geq 0$, one has necessarily $0 \leq Y, Z \leq X$, $0 \leq D = d - Y \leq d$, hence $d \to 0$ implies $D \to 0$, $Z \to X$, $Y \to 0$. In this situation the limiting CAPM based fair premium coincides necessarily with the *limiting CAPM based mean risk premium.* To illustrate the CAPM based rate-making method, we specialize in the following to the main and simplest example of a stop-loss reinsurance $Z = (X-d)_+$, $D = (d-X)_+$. In the usual notations a calculation shows that

$$P = P(d) = \mu + \frac{\chi(d)}{\tau^2(d) + \pi(d)\chi(d)} \sigma^2 , \tag{4.19}$$

$$\overline{P} = \overline{P}(d) = P(d) - \chi(d) = \mu + \frac{\sigma^2(d) + \pi(d)\chi(d)}{\tau^2(d) + \pi(d)\chi(d)} \chi(d) . \tag{4.20}$$

Theorem 4.1. Let $Z = (X-d)_+$, $D = (d-X)_+$, and assume the risk X takes non-negative values. Then the limiting CAPM based fair premium is equal to

$$\begin{aligned} P_{\lim} &= \lim_{D \to 0} \left\{ E[X] + \frac{\mathrm{Var}[X]}{\mathrm{Cov}[X,Y]} E[D] \right\} = (1+k^2)\mu , \qquad \text{with} \\ k &= \frac{\sigma}{\mu} \qquad \text{the coefficient of variation.} \end{aligned} \tag{4.21}$$

Proof. From (4.19) one has

$$P_{\lim} = \lim_{d \to 0} \left\{ \mu + \frac{\sigma^2}{\frac{\tau^2(d)}{\chi(d)} + \pi(d)} \right\} .$$

Since $\pi(0) = \mu$ it suffices to show that $\lim_{d \to 0} \frac{\tau^2(d)}{\chi(d)} = 0$. But one has

$$\frac{\tau^2(d)}{\chi(d)} = \frac{\mathrm{Var}[Y]}{E[D]} = \frac{\mathrm{Var}[D]}{E[D]} = \frac{E[(d-X)_+^2]}{E[(d-X)_+]} - E[(d-X)_+] .$$

Since $X \geq 0$ the second term goes to zero as d goes to zero. Using that $0 \leq (d-X)_+ \leq d$, the first term satisfies the inequality

$$0 \leq \frac{E[(d-X)_+^2]}{E[(d-X)_+]} \leq d\,.$$

The result follows.

Remark 4.1. The limiting CAPM based fair premium identifies with the distribution-free and parameter-free modified variance principle first advocated by Heilmann (1988), which finds herewith an insurance economics interpretation. In case of arbitrary risks taking values on the whole real line, the random variable $D = (d-X)_+$ is unbounded, and the limiting behaviour of the formulas (4.16), (4.18) is quite different.

Example 4.3. Suppose X is a random variable following Bowers' distribution (1.5). Then one has

$$\pi(d)\chi(d) = \frac{1}{4}\sigma^2, \quad \sigma(d) = \tau(d) = \frac{1}{2}\sigma$$

uniformly for all deductibles d. It follows that

$$P(d) = \mu + 2\chi(d) = d + \sqrt{(d-\mu)^2 + \sigma^2}\,, \tag{4.22}$$

$$\overline{P}(d) = \mu + \chi(d) = \frac{1}{2}(\mu + d + \sqrt{(d-\mu)^2 + \sigma^2})\,. \tag{4.23}$$

In the limiting case as d goes to zero, one gets for $\mu \geq 0$

$$P(0) = \mu\sqrt{1+k^2}, \quad \overline{P}(0) = \frac{1}{2}(1 + \sqrt{1+k^2})\mu < P(0)\,. \tag{4.24}$$

Since X may take negative values, as d goes to zero, the guaranteed dividend $D = (-X)_+$ does not vanish. Observe that in practice arbitrary risks occur in Finance (e.g. yield valuation of stocks) as well as in Insurance (e.g. mixed portfolio valuations of whole life insurances and life annuities). Furthermore it is striking to note that $\overline{P}(\mu) = \mu + \frac{1}{2}\sigma$ coincides with the insurance version of the best upper bound (4.5) in Hürlimann (1991) obtained using another financial risk model, a fact which must be emphasized in view of possible misinterpretations.

Remark 4.2. Numerical examples suggest the existence of an "optimal" deductible (which presumably lies below but close to the mean) for which the mean risk premium is maximum. In this situation the mean risk premium charges the largest security loading, and defines the safest possible CAPM based model design. However for competitive reasons it seems that only those deductibles should be considered for which $\overline{P}(d) \le \overline{P}(0)$. In case $X \ge 0$ numerical examples suggest that there exist $d_0 \ge \mu$ such that $\overline{P}(d) \le \overline{P}(0) = (1+k^2)\mu$ for all $d \ge d_0$ and $\overline{P}(d)$ is monotone decreasing on $[d_0, \infty)$. Suppose now that $P = (1+k^2)\mu$ is the market price for covering the risk X. Then the unique deductible $d \ge d_0$ for which $\overline{P}(d) = (1+k^2)\mu$ leads in the mean to the same expected surplus for the insurer as traditional insurance. In the latter situation this expected surplus is guaranteed in the mean, and the realized surplus is greater than the expected surplus provided realized claims are in the average less than the expected claims. As simulation runs show (see example 4.5) the stop-loss experience rating scheme with fair premium $P(d)$ and dividend $D = (d-X)_+$ can leave to the insurer a smaller realized surplus when realized claims are in the average less than the mean. However in this case the simulated value $P(d) - D$ of the CAPM based mean risk premium is closer to the realized claims than for traditional insurance, which results in increased fairness.

For practical purposes it may be useful to have more handy but still general lower and upper bounds for the CAPM based risk premiums (4.19), (4.20). These bounds follow from a slightly generalized version of the inequality of Kremer (1990) (see also Hürlimann (1994a)).

Proposition 4.2. In the usual notations the following inequalities hold:

$$\frac{F(d)}{\overline{F}(d)}\pi(d)^2 \le \sigma^2(d) \le \sigma^2 - 2\pi(d)\chi(d) - \frac{\overline{F}(d)}{F(d)}\chi(d)^2 , \tag{4.25}$$

$$\frac{\overline{F}(d)}{F(d)}\chi(d)^2 \le \tau^2(d) \le \sigma^2 - 2\pi(d)\chi(d) - \frac{F(d)}{\overline{F}(d)}\pi(d)^2 . \tag{4.26}$$

Proof. Conditioning on the event $\{X > d\}$ respectively $\{X < d\}$ one gets successively:

$$\begin{aligned}
\sigma^2(d) + \pi(d)^2 &= E[(X-d)_+^2] \\
&= \overline{F}(d)E[(X-d)^2 \mid X > d] \\
&= E[(X-d)^2] - F(d)E[(X-d)^2 \mid X \le d] \\
&= \sigma^2 + (d-\mu)^2 - F(d)E[(X-d)^2 \mid X \le d] .
\end{aligned}$$

Now insert the inequality

$$E[X-d \mid X \leq d]^2 \leq E[(X-d)^2 \mid X \leq d]$$

into the above expressions and use the relations

$$\chi(d) = -F(d)E[X-d \mid X \leq d]\,, \qquad d-\mu = \chi(d) - \pi(d)$$

to get the upper bound in (4.25). For the lower bound one proceeds similarly:

$$\begin{aligned}\sigma^2(d) &= \overline{F}(d)E[(X-d)^2 \mid X > d] - \pi(d)^2 \\ &\geq \overline{F}(d)E[X-d \mid X > d]^2 - \pi(d)^2 \\ &= \frac{F(d)}{\overline{F}(d)}\pi(d)^2\,.\end{aligned}$$

The inequalities (4.26) follow from (4.25) and the relation $\sigma^2(d) + \tau^2(d) = \sigma^2 - 2\pi(d)\chi(d)$.

Since $P(d) = \overline{P}(d) + \chi(d)$ it suffices to formulate bounds for $\overline{P}(d)$ only.

Corollary 4.1. Let $Z = (X-d)_+$ define an experience rated insurance contract with guaranteed dividend $D = (d-X)_+$. Then the variance factor loading of the corresponding CAPM based mean risk premium satisfies the following inequalities

$$\begin{aligned}\frac{\chi(d)}{\sigma^2 - \pi(d)\chi(d)} &\leq \frac{\chi(d)}{\sigma^2 - \pi(d)\chi(d) - \frac{F(d)}{\overline{F}(d)}\pi(d)^2} \\ &\leq \frac{\overline{P}(d)-\mu}{\sigma^2} \qquad\qquad (4.27)\\ &\leq \frac{1}{\pi(d) + \frac{\overline{F}(d)}{F(d)}\chi(d)^2} \leq \frac{1}{\pi(d)}\,.\end{aligned}$$

In numerous cases of practical interest exact formulas for the above CAPM based premium rating theory may be given analytically or evaluated numerically.

Example 4.4. Let X be normally distributed with mean μ and variance σ^2. Let $N(x)$ be the distribution of the standard normal variate and let $\Phi(x) = N'(x)$ be the normal density. Setting

$$\beta = \beta(d) = \frac{\mu - d}{\sigma}\,,$$

a straightforward calculation leads to the formulas

$$\begin{aligned}
\pi(d) &= \sigma[\Phi(\beta) + \beta N(\beta)], \\
\chi(d) &= \sigma[\Phi(\beta) - \beta(1 - N(\beta))], \\
\sigma^2(d) + \pi(d)\chi(d) &= \sigma^2 N(\beta), \\
\tau^2(d) + \pi(d)\chi(d) &= \sigma^2[1 - N(\beta)].
\end{aligned} \tag{4.28}$$

It follows that

$$\overline{P}(d) = \mu + \left(\frac{N(\beta)}{1 - N(\beta)}\right)[\Phi(\beta) - \beta(1 - N(\beta))]\sigma. \tag{4.29}$$

In particular one has

$$\overline{P}(\mu) = \mu + \frac{\sigma}{\sqrt{2\pi}}, \tag{4.30}$$

which coincides with the insurance version of the "realistic" approximation (4.3) in Hürlimann (1991) derived on the basis of another model (with different interpretation). Table 2 gives a numerical illustration in case $\mu = 100$, $\sigma = 10$. One sees that $\overline{P}(d)$ has a maximum at approximately $d = 90$ and $\overline{P}(d) \leq (1 + k^2)\mu = 101$ for $d \leq 30$ and $d \geq 120$.

Table 2: mean risk premium for a normally distributed risk

d	$F(d)$	$\pi(d)$	$H[Z]$	$P(d)$	$\overline{P}(d)$
20	0.000	80.000	80.000	100.000	100.000
30	0.000	70.000	70.914	100.914	100.914
40	0.000	60.000	61.365	101.365	101.365
50	0.000	50.000	51.783	101.783	101.783
60	0.000	40.000	42.236	102.236	102.236
70	0.001	30.004	32.829	102.829	102.825
80	0.023	20.085	23.732	103.732	103.647
90	0.159	10.833	15.251	105.251	104.418
100	0.500	3.989	7.979	107.979	103.989
110	0.841	0.833	2.876	112.876	102.043
120	0.977	0.085	0.552	120.552	100.468
130	0.999	0.004	0.044	130.044	100.041
140	1.000	0.000	0.001	140.001	100.001
150	1.000	0.000	0.000	150.000	100.000

Example 4.5. Let X represent the aggregate claims of a whole life insurance portfolio, which consists of $3'216$ policies with characteristic figures $\mu = 1'612'710$, $\sigma = 564'288$. The numerical results of a simulation with 10 samples are reproduced in Table 3.

Table 3: simulated mean risk premium for a whole life insurance portfolio

d	$F(d)$	$\pi(d)$	$H[Z]$	$P(d)$	$\overline{P}(d)$	average of $P(d) - D$
0	0.00002	1 612 712	1 810 112	1 810 112	1 810 112	1 810 112
500 000	0.01373	1 113 692	1 355 906	1 855 910	1 854 924	1 855 906
1 000 000	0.15776	640 815	942 783	1 942 783	1 914 678	1 932 783
1 500 000	0.48673	281 343	582 137	2 082 137	1 913 504	1 842 137
2 000 000	0.79009	90 738	301 913	2 301 913	1 823 884	1 691 913
2 500 000	0.94063	21 672	123 195	2 623 195	1 714 234	1 563 195
3 000 000	0.98779	3 939	37 571	3 037 571	1 646 342	1 477 571

A more detailed numerical output shows that $\overline{P}(d)$ is maximum at $d = 1\,300\,000$ and that $\overline{P}(d) \leq (1 + k^2)\mu$ for $d \geq 2\,100\,000$.

5 Diatomic safe approximations to mean risk premiums

For notations we refer to the preceding Sections. For actuarial purposes, e.g. if Z is a reinsurance, it may be desirable to have a (relative) safe estimate for $H[Z]$. This can be achieved taking the diatomic risk Z^* with maximum expected value over all diatomic risks:

$$E[Z^*] = \max_{X \in D_2[a,b]} E[v(X)]\,. \tag{5.1}$$

Let $\{x_1 = x, x_2 = y\}$ be the support of the maximizing diatomic distribution. To calculate it consider the stationary point of the Lagrange function

$$L(x, y, \lambda) = \left(\frac{y-\mu}{y-x}\right)v(x) + \left(\frac{\mu - x}{y-x}\right)v(y) + \lambda(\sigma^2 - (\mu - x)(y-\mu))\,. \tag{5.2}$$

A calculation shows that $L_x = L_y = L_\lambda = 0$ is satisfied provided $\{x, y\}$ is solution of the pair of equations

$$\begin{aligned} \frac{v(y) - v(x)}{y - x} &= \frac{1}{2}(v'(x) + v'(y))\,, \\ (\mu - x)(y - \mu) &= \sigma^2\,. \end{aligned} \tag{5.3}$$

To guarantee a maximum, one must verify that the Hessian of L is negative semidefinite and that $x, y \in [a, b]$. If $x \leq a$ or $y \geq b$, or the Hessian of L is not negative semidefinite, the maximum is attained on the boundary of the domain of variation for x, y. Simplification in the notation is obtained by considering the (algebraical) involution on the interval $[a, b]$, which maps x to

$$x^* = \mu + \frac{\sigma^2}{\mu - x}. \tag{5.4}$$

The solution to the optimization problem (5.1) may be summarized as follows.

Lemma 5.1. Let $Z = v(X)$ be a transformed random variable of $X \in D_2[a, b]$, and let $Z^* = v(X^*)$, where $X^* \in D_2[a, b]$ has support $\{x, x^*\}$, be the maximizing diatomic distribution defined by (5.1). Then one of the following conditions holds:

(i) $x, x^* \in (a, b)$ is solution of the equation

$$\frac{v(x^*) - v(x)}{x^* - x} = \frac{1}{2}(v'(x) + v'(x^*)) \tag{5.5}$$

(ii) $x = a, \quad x^* = a^*$

(iii) $x = b^*, \quad x^* = b^{**} = b$

Example 5.1. If $Z = v(X) = (X - d)_+$ it suffices to consider $x \leq d \leq x^*$. One has $v(x) = 0$, $v'(x) = 0$, $v(x^*) = x^* - d$, $v'(x^*) = 1$. Solving (5.5) one gets

$$\begin{aligned} x &= d - \sqrt{(d-\mu)^2 + \sigma^2}, \\ x^* &= d + \sqrt{(d-\mu)^2 + \sigma^2}. \end{aligned} \tag{5.6}$$

The maximum

$$E[Z^*] = \left(\frac{\mu - x}{x^* - x}\right)(x^* - d) \tag{5.7}$$

is attained as follows:

Case 1:	$a \leq d \leq \frac{1}{2}(a + a^*)$,	$x = a$
Case 2:	$\frac{1}{2}(a + a^*) \leq d \leq \frac{1}{2}(b + b^*)$,	$x = d - \sqrt{(d-\mu)^2 + \sigma^2}$
Case 3:	$\frac{1}{2}(b + b^*) \leq d \leq b$,	$x = b^*$

Note that (5.7) identifies actually with the best upper bound for the net stop-loss premium $E[Z]$ over all risks defined on $[a,b]$ with given mean and variance. This result can be found in several actuarial publications, e.g. De Vylder and Goovaerts (1982), Goovaerts et al. (1984), p. 316, Jansen et al. (1986), Goovaerts et al. (1990). The important special case of positive risks $X \geq 0$ is solved by the limiting case $a = 0$, $b \to \infty$, for which only case 1 and case 2 occur. The special case $a \to -\infty$, $b \to \infty$, is the well-known inequality of Bowers (1969), recovered through case 2.

Let now X be a risk with support $[a,b]$ and with variance premium $P = H[X] = \mu + \theta\sigma^2$. For rate-making purposes let us consider besides the risk X the diatomic risk X^* given by the extremal solution $X^* = \{x, x^*\}$ to (5.1). Similarly besides the splitting risk components $Y = u(X)$, $Z = v(X)$ such that $X = Y + Z$, consider the diatomic splitting components $Y^* = u(X^*)$, $Z^* = v(X^*)$ such that $X^* = Y^* + Z^*$. Since X^* has the same mean and variance as X, its variance premium is equal to $H[X^*] = H[X] = P$. Moreover the CAPM splitting risk premiums (4.14) are equal to

$$H[Y^*] = u(x) + \left(\frac{u(x^*) - u(x)}{x^* - x}\right)(P - x)\,, \tag{5.8}$$

$$H[Z^*] = v(x) + \left(\frac{v(x^*) - v(x)}{x^* - x}\right)(P - x)\,. \tag{5.9}$$

The CAPM splitting risk premium system defined by $H^*[X] := H[X^*]$, $H^*[Y] := H[Y^*]$, $H^*[Z] := H[Z^*]$ clearly satisfies the splitting property $P = H^*[X] = H^*[Y] + H^*[Z]$. It is a diatomic approximation to the CAPM splitting risk premium system $H[X]$, $H[Y]$, $H[Z]$ such that $P = H[X] = H[Y] + H[Z]$, where the splitting risk premiums $H[Y]$, $H[Z]$ are defined by (4.14). On the other side the maximum (5.1), denoted by say $\pi^*[Z] = E[Z^*]$, is given by

$$\pi^*[Z] = v(x) + \left(\frac{v(x^*) - v(x)}{x^* - x}\right)(\mu - x)\,. \tag{5.10}$$

Since $P \geq \mu$ one sees that $H^*[Z] = H[Z^*] \geq \pi^*[Z]$. The following natural generalized refinement of the insurance market based distribution-free stop-loss premium principle first derived in Hürlimann (1993) holds.

Theorem 5.1. Given is a insurance risk X with finite mean μ and variance σ^2, which takes values in the interval $[a,b]$. Let $X = Y + Z$, $Y = u(X)$, $Z = v(X)$, be a splitting of X, and let $Z^* = v(X^*)$ be a solution of the otimization problem

(5.1). Then the CAPM splitting risk premium system $H^*[X] = H[X^*] = P$, $H^*[Y] = H[Y^*]$, $H^*[Z] = H[Z^*]$ defined by (5.8), (5.9), satisfies the splitting property $P = H^*[X] = H^*[Y] + H^*[Z]$ as well as the criterion of safeness

$$H^*[Z] \geq \pi^*[Z] = \max_{X \in D_2[a,b]} E[v(X)] . \tag{5.11}$$

In the experience rating framework of Section 4, let us restrict now the attention to feasible reinsurance contracts with maximum deductible d such that $d + Z = X + D$. The advocated condition of immunization $H^*[Y] = d$ allows to solve for a CAPM based distribution-free fair premium:

$$P = x + \left(\frac{x^* - x}{u(x^*) - u(x)} \right) d(x) , \qquad \text{with} \quad d(x) = d - u(x) . \tag{5.12}$$

The associated CAPM based distribution-free reinsurance premium can be written as

$$H^*[Z] = v(x) + \left(\frac{v(x^*) - v(x)}{u(x^*) - u(x)} \right) d(x) , \tag{5.13}$$

and the corresponding CAPM distribution-free mean risk premium diatomic approximation is given by

$$\overline{P}^* = \mu - \pi^*[Z] + H^*[Z] . \tag{5.14}$$

The practical importance of the above distribution-free rate making method is best illustrated at the special case of stop-loss reinsurance $v(X) = (X - d)_+$, $D(X) = (d - X)_+$, in the frequently encountered instance of positive risks $X \geq 0$, hence $a = 0, b = \infty$. Since $x \leq d \leq x^*$ one has $v(x) = 0, v(x^*) = x^* - d$, $u(x) = x$, $u(x^*) = d$, $d(x) = d - x$. Depending on the deductible d the associated fair risk premium diatomic approximation is $P^*(d) = x^*$. From the point of view of the insured, the corresponding mean risk premium is given by $\overline{P}(d) = P^*(d) - \chi(d)$, which is different from the diatomic approximation $\overline{P}^*(d) = P^*(d) - \chi^*(d)$ as in (5.14). From example 5.1 one sees that two cases must be distinguished:

$$\text{case 1:} \quad \overline{P}(d) = (1 + k^2)\mu - \chi(d) , \quad 0 \leq d \leq \frac{1}{2}(1 + k^2)\mu , \quad k = \frac{\sigma}{\mu} \tag{5.15}$$

$$\text{case 2:} \quad \overline{P}(d) = d + \sqrt{(d - \mu)^2 + \sigma^2} - \chi(d), \quad d \geq \frac{1}{2}(1 + k^2)\mu \tag{5.16}$$

In general the mean risk premium depends via $\chi(d) = d - \mu + \pi(d)$ on the net stop-loss premium, hence on the distribution of the risk $F(x) = \Pr(X \leq x)$.

Let us determine the least possible mean risk premium $\overline{P}_{\min} = \inf_{d\geq 0}\{\overline{P}(d)\}$. The result depends on the coefficient of variation k and on the behaviour of the function $\overline{P}(d)$ in the interval $I = [\frac{1}{2}(1+k^2)\mu, \mu]$. The open interval $(\frac{1}{2}(1+k^2)\mu, \mu)$ is denoted by I^O. The distribution of Bowers (1.5) is denoted by $F^B(x)$.

Theorem 5.2. Let $Z = (X-d)_+$ be the stop-loss claim with deductible d associated to a positive risk $X \geq 0$. Then the minimum value of the mean risk premium (5.15), (5.16) can be calculated as follows:
Case I. If $k \geq 1$ then one has

$$\overline{P}_{\min} = \overline{P}\left(\frac{1}{2}(1+k^2)\mu\right) = (1+k^2)\mu - \chi\left(\frac{1}{2}(1+k^2)\mu\right) \tag{5.17}$$

Case II. If $k < 1$ then three subcases are possible:

(II.a) If $F(x) \leq 2F^B(x)$ for all $x \varepsilon I$, then one has

$$\overline{P}_{\min} = \overline{P}\left(\frac{1}{2}(1+k^2)\mu\right) \tag{5.18}$$

(II.b) If $F(x) \geq 2F^B(x)$ for all $x \varepsilon I$, then one has

$$\overline{P}_{\min} = \overline{P}(\mu) = (1+k^2)\mu - \pi(\mu) \tag{5.19}$$

(II.c) If $F(x) = 2F^B(x)$ for some $d_1, \ldots, d_m \varepsilon I^O$, then one has

$$\overline{P}_{\min} = \min\left\{\overline{P}\left(\frac{1}{2}(1+k^2)\mu\right), \overline{P}(d_1), \ldots, \overline{P}(d_m), \overline{P}(\mu)\right\} \tag{5.20}$$

Proof. In case 1 one has $\overline{P}'(d) = -F(d) < 0$, which shows that $\overline{P}(d)$ is strictly monotone decreasing on the interval $[0, \frac{1}{2}(1+k^2)\mu]$. In case 2 one has

$$\overline{P}'(d) = 2F^B(d) - F(d)\,. \tag{5.21}$$

Now if $k \geq 1$ and $d \geq \frac{1}{2}(1+k^2)\mu$ one gets

$$\begin{aligned}\overline{P}'(d) \geq 2F^B(d) - 1 \geq 2F^B&\left(\frac{1}{2}(1+k^2)\mu\right) - 1 \\ &= 2\frac{k^2}{k^2+1} - 1 \\ &= \frac{k^2-1}{k^2+1} \geq 0\,,\end{aligned} \tag{5.22}$$

which shows that $\overline{P}(d)$ is monotone increasing on the interval $[\frac{1}{2}(1+k^2)\mu, \infty)$. Therefore (5.17) holds. If $k < 1$ then the inequality

$$\overline{P}'(d) \geq 2F^B(d) - 1 \geq 2F^B(\mu) - 1 = 0\,, \qquad \text{for all} \quad d \geq \mu\,, \tag{5.23}$$

shows that $\overline{P}_{\min}$ is attained on the closed interval I. The further details follow from elementary calculus. Figure 1 shows graphically what may happen.

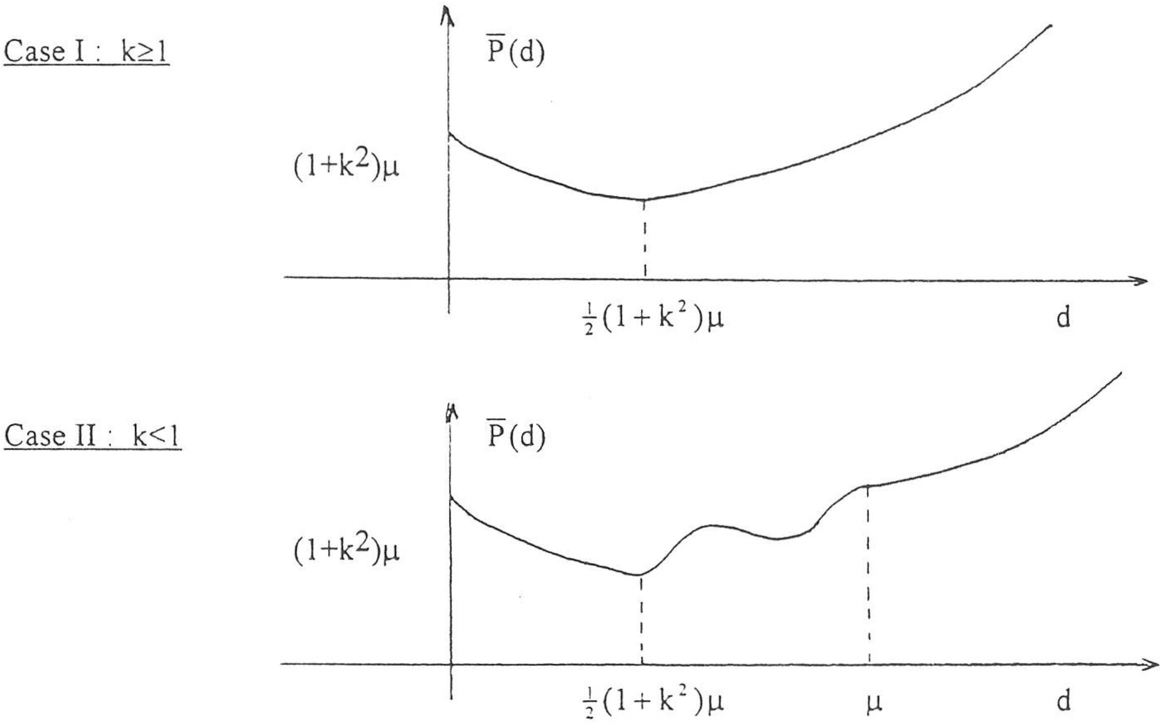


Figure 1: dependence of the mean risk premium on the deductible

In case the distribution $F(x) = \Pr(X \leq x)$ is not known and only information about μ, σ is available, let us determine the following "confidence interval":

$$\inf_{d \geq 0} \left\{ \inf_{X \in D^+} \{\overline{P}(d)\} \right\} \leq \overline{P}_{\min} \leq \inf_{d \geq 0} \left\{ \sup_{X \in D^+} \{\overline{P}(d)\} \right\}, \tag{5.24}$$

where D^+ denotes the set of all random variables taking values in $[0, \infty)$ with fixed μ and σ.

Theorem 5.3. Let $X \geq 0$ be a positive risk with mean μ and variance σ^2. Then one has

$$\overline{P}_{\inf}(d) := \inf_{X \in D^+} \{\overline{P}(d)\} = \begin{cases} (1+k^2)\mu - \left(\frac{k^2}{k^2+1}\right)d\,, & 0 \leq d \leq \frac{1}{2}(1+k^2)\mu \\ \frac{1}{2}\left(d+\mu+\sqrt{(d-\mu)^2+\sigma^2}\right), & d \geq \frac{1}{2}(1+k)^2\mu \end{cases} \tag{5.25}$$

The value of $\overline{P}_{\sup}(d) = \sup_{X \in D^+}\{\overline{P}(d)\}$ depends on the coefficient of variation as follows. If $k \geq 1$ then one has

$$\overline{P}_{\sup}(d) = \begin{cases} (1+k^2)\mu, & 0 \leq d \leq \mu \\ (1+k^2)\mu - (d-\mu), & \mu \leq d \leq \frac{1}{2}(1+k^2)\mu \\ \mu + \sqrt{(d-\mu)^2+\sigma^2}, & d \geq \frac{1}{2}(1+k^2)\mu \end{cases} \tag{5.26}$$

If $k < 1$ then one has

$$\overline{P}_{\sup}(d) = \begin{cases} (1+k^2)\mu, & 0 \leq d \leq \frac{1}{2}(1+k^2)\mu \\ d + \sqrt{(d-\mu)^2+\sigma^2}, & \frac{1}{2}(1+k^2)\mu \leq d \leq \mu \\ \mu + \sqrt{(d-\mu)^2}+\sigma^2, & d \geq \mu \end{cases} \tag{5.27}$$

Furthermore the lower bound

$$\inf_{d \geq 0}\{\overline{P}_{\inf}(d)\} = \left(1+\frac{1}{2}k^2\right)\mu \tag{5.28}$$

is attained at $d = \frac{1}{2}(1+k^2)\mu$ and the upper bound

$$\inf_{d \geq 0}\{\overline{P}_{\sup}(d)\} = \begin{cases} \left(1+\frac{1}{2}k^2\right)\mu + \frac{1}{2}\mu, & k \geq 1 \\ (1+k^2)\mu, & k < 1 \end{cases}$$

is attained at $d = \frac{1}{2}(1+k^2)\mu$ in case $k \geq 1$ and at all $0 \leq d \leq \frac{1}{2}(1+k^2)\mu$ in case $k < 1$.

Proof. From Goovaerts et al. (1984), p. 316, one gets easily

$$\sup_{X \in D+} \{\chi(d)\} = d - \mu + \sup_{X \in D+} \{\pi(d)\}$$

$$= \begin{cases} \left(\frac{k^2}{k^2+1}\right)d, & 0 \le d \le \frac{1}{2}(1+k^2)\mu \\ \frac{1}{2}(\sqrt{(d-\mu)^2+\sigma^2} + (d-\mu)), & d \ge \frac{1}{2}(1+k^2)\mu \end{cases} \tag{5.30}$$

$$\inf_{X \in D+} \{\chi(d)\} = \begin{cases} 0, & 0 \le d \le \mu \\ d-\mu, & d \ge \mu \end{cases} \tag{5.31}$$

from which one obtains successively (5.25), (5.26), (5.27). Calculation of derivatives yield

$$\overline{P}'_{\inf}(d) = \begin{cases} -\left(\frac{k^2}{k^2+1}\right) < 0, & 0 \le d \le \frac{1}{2}(1+k^2)\mu \\ F^B(d) > 0, & d \ge \frac{1}{2}(1+k^2)\mu \end{cases}$$

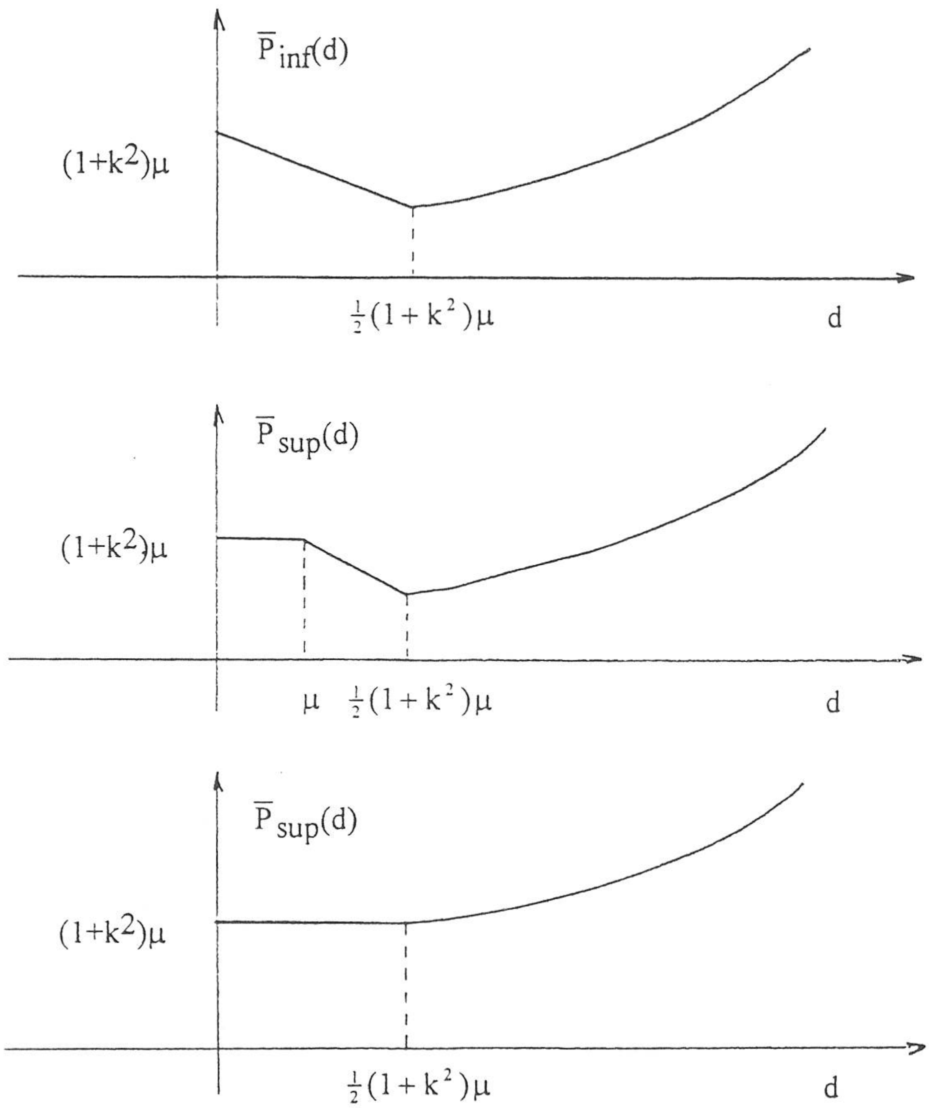
which shows that $\overline{P}_{\inf}(d)$ is minimum at $d = \frac{1}{2}(1+k^2)\mu$. The result about $\overline{P}_{\sup}(d)$ is shown similarly. Figure 2 provides a graphical picture of the situation.

Remark 5.1.

(i) The uniquely defined "optimal" deductible $d = \frac{1}{2}(1+k^2)\mu$ of Theorem 5.3 has been obtained first in Hürlimann (1994a), Section 4.2, under more awkward assumptions. But this first study has served as a guide towards the present "best" confidence interval for the mean risk premium.

(ii) It is further instructive to observe that for an unbounded positive risk, one has $H^*[Z] = \sqrt{(d-\mu)^2+\sigma^2}$ for $d \ge \frac{1}{2}(1+k^2)\mu$, which says that the CAPM based distribution-free stop-loss premium goes to infinity as d goes to infinity. Of course for a bounded risk taking values in a finite interval $[a, b]$, the results must be adapted taking into account case 3 of example 5.1.

(iii) In view of the above results, statistical estimation procedures for the coefficient of variation are of practical interest. In the normally distributed case, the "old" approximation by Hendricks and Robey (1936) has been shown to be a uniform approximation to the exact sampling distribution of the coefficient of variation with an exponential error bound of magnitude $\exp(-\frac{n}{2k^2})$, where n is the sample size (see Hürlimann (1994d)).



Case $k \geq 1$

Case $k < 1$

Figure 2: distribution-free lower and upper bound for the mean risk premium

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Summary

The effect on premium calculation of splitting a risk into two components is analyzed. Provided the total splitting risk is measured by the variance, the maximum variance premium reduction is attained for a linear risk-exchange, in which the mean-level of the retained risk can be chosen and half of the mean claims deviation is exchanged. If the total splitting risk is measured by the standard deviation, no standard deviation premium reduction can be achieved. Status quo is reached with a linear risk-exchange. Then the design of a premium calculation principle, which is additive for independent risks and for splitting risk components, is discussed. It is shown in a special case that a premium principle with these additive properties, and characterized by the values it takes on the set of diatomic risks with given mean and variance, necessarily satisfies the CAPM relations by Borch (1982). Some consequences for premium calculation are derived in the special case a splitting component belongs to a well-defined class of feasible reinsurance contracts with a fixed maximum deductible. Such a contract induces an experience rated insurance contract, which offers a perfectly hedged bonus. This immunization property and the CAPM relationships lead to parameter-free CAPM based premium formulas. Notions of fair premium and mean risk premium are introduced. In the special case of stop-loss reinsurance and non-negative risks, the limiting CAPM based fair premium as the bonus goes to zero identifies with the modified variance premium advocated by Heilmann (1988), and which finds herewith an insurance economics interpretation. Finally interesting distribution-free and rather robust results are obtained if the CAPM based reinsurance premium is replaced by a safe diatomic estimate.

Zusammenfassung

Der Einfluss, der eine Risikozerlegung in zwei Komponenten auf die Prämienberechnung ausübt, wird analysiert. Falls das gesamte Zerlegungsrisiko anhand der Varianz gemessen wird, so ergibt sich eine maximale Varianzprämienreduktion für einen linearen Risikoaustausch, der das mittlere Selbstbehaltsrisiko zur Auswahl gibt, und der die Hälfte der effektiven Abweichung vom erwarteten Risiko zum Austausch vorschreibt. Falls das gesamte Zerlegungsrisiko anhand der Standardabweichung gemessen wird, so ist keine Standardabweichungsprämienreduktion möglich. Das Status quo wird für einen linearen Risikoaustausch erreicht. Anschliessend wird die Konstruktion eines Prämienprinzips diskutiert, das additiv für unabhängige Risiken ist und additiv auf die Risikozerlegungskomponenten wirkt. Es wird in einem Spezialfall gezeigt, dass ein Prämienprinzip mit diesen additiven Zerlegungseigenschaften, das durch die genommenen Werte auf die Menge der zweipünktigen Risiken mit gegebenen Erwartungswert und Varianz charakterisiert wird, notwendigerweise die CAPM Verknüpfungen von Borch (1982) erfüllt. Es werden einige Folgerungen für die Prämienberechnung für den Spezialfall abgeleitet, dass eine Risikozerlegungskomponente Element einer wohldefinierten Klasse von Rückversicherungsverträgen mit fester maximaler Franchise ist. Ein solcher Vertrag induziert eine Erfahrungstarifierung, die einen vollständig abgesicherten Bonus zur Folge hat. Diese Immunisierungseigenschaften und die CAPM Verknüpfungen ermöglichen es, parameterfreie CAPM Prämienformeln herzuleiten. Die Begriffe einer fairen Prämie und einer mittleren Risikoprämie werden eingeführt. Im speziellen Fall einer Stop-Loss Rückversicherung und von nicht-negativen Risiken wird die CAPM faire Prämiengrenze, wenn der Bonus gegen Null strebt, mit der modifzierten Varianzprämie von Heilmann (1988) identifiziert, was eine versicherungsökonomische Interpretation dieses Prinzips liefert. Schliesslich werden interessante verteilungsfreie und ziemlich robuste Ergebnisse für den Fall vorgestellt, das die CAPM Rückversicherungsprämie durch eine sichere zweipünktige Schätzung ersetzt wird.

Résumé

L'effet induit sur le calcul des primes par décomposition d'un risque en deux composantes est analysé. Si le risque de décomposition total est mesuré par la variance, la réduction en prime de variance maximale est atteinte avec un échange de risque linéaire, pour lequel le risque retenu moyen peut être choisi et la moité de l'écart effectif par rapport aux sinistres moyens est échangé. Si le risque de décomposition total est mesuré par l'écart-type, une réduction de la prime d'écart-type n'est pas possible. Le status quo est atteint pour un échange de risque linéaire. Ensuite on discute la construction d'un principe de calcul des primes, qui est additif pour des risques indépendants et qui agit additivement sur les composantes du risque. Il est montré dans un cas particulier qu'un principe de calcul des primes satisfaisant ces propriétés additives, et caractérisé par les valeurs qu'il prend sur l'ensemble des risques diatomiques de moyenne et variance donnés, nécessairement vérifie les relations CAPM de Borch (1982). On obtient quelques conséquences pour le calcul des primes dans le cas particulier d'une composante de risque appartenant à une classe bien définie de contrats de réassurance de franchise maximale donnée. Un tel contrat induit un contrat d'assurance par expérience, qui offre une participation aux bénéfices parfaitement couverte. Cette propriété d'immunisation et les relations CAPM permettent d'obtenir des formules de primes CAPM libres de paramètres. Des notions de prime juste et de prime de risque moyenne sont introduites. Dans le cas particulier d'une réassurance stop-loss et de risque non-négatifs, la prime juste CAPM limite lorsque la participation aux bénéfices tend vers zéro s'identifie avec la prime de variance modifiée introduite par Heilmann (1988), qui trouve ainsi une interprétation économique. Finalement on obtient des résultats libres d'hypothèse sur la fonction de répartition des sinistres et assez robustes lorsque la prime de réassurance CAPM est remplacée par une estimation diatomique sûre.