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D. Kurzmitteilungen

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The p-th power variance principle

1 Introduction

In the terminology of Bühlmann (1971, p. 85), Goovaerts, de Vylder and Haezendock (1984, p. 16), and Heilmann (1987, p. 110), a principle of premium calculation H is a functional that assigns a number $P = H(S)$, the premium, to any given risk S, a random variable. Examples are the *variance principle,* where

$$H(S) = E(S) + \alpha\,\mathrm{Var}(S)\,, \qquad \alpha > 0$$

and the *standard deviation principle,*

$$H(S) = E(S) + \beta\sqrt{\mathrm{Var}(S)}, \qquad \beta > 0\,.$$

In this note, we shall imbed these two principles in a one parameter family. We consider more generally the *p-th power variance principle,* where

$$H(S) = E(S) + \gamma\,\mathrm{Var}(S)^p, \qquad \gamma > 0, \quad p \geq 0.5\,.$$

The condition $p \geq 0.5$ is necessary to assure convexity of the p-th power variance principle in the sense of Deprez and Gerber (1985).

2 The problem of optimal cooperation

As in Gerber (1980, p. 78), we suppose that each of n companies determines the premium according to a given principle. Let $H_i(\cdot)$ denote the principle of company i, $i = 1, 2, \cdots, n$. Also, let S be a given risk to be insured by the n companies. How can the risk S be decomposed so that the total premium is a minimum? Mathematically, this is the problem of minimising

$$\sum_{i=1}^{n} H_i(S_i)\,,$$

where $S_1, S_2, \ldots, S_n$ are random variables such that $S = S_1 + S_2 + \cdots + S_n$.

If all the companies use a variance principle (company i with parameter α_i, $i = 1, 2, \ldots, n$), it has been shown that this problem has an explicit solution. The premium is minimal for

$$S_i = \frac{\alpha_i}{\alpha} S\,,$$

with

$$\frac{1}{\alpha} = \sum_{i=1}^{n} \frac{1}{\alpha_i}\,,$$

and its value is

$$E(S) + \alpha\,\mathrm{Var}(S)\,.$$

This result has an intuitive interpretation. It is the premium that would be charged by a single company that uses the variance principle with parameter α. The goal of this note is to generalise these results to the p-th power variance principle.

3 Proportional decompositions

We suppose that company i uses the p-th power variance principle with parameters γ_i and $p > 0.5$. Thus for a decomposition $S_1, S_2, \ldots, S_n$ of S, the total premium is

$$\sum_{i=1}^{n} H_i(S_i) = E(S) + \sum_{i=1}^{n} \gamma_i\,\mathrm{Var}(S_i)^p\,.$$

A first step is to consider proportional decompositions of S, i.e.,

$$S_i = f_i S\,,$$

where $\sum_{i=1}^{n} f_i = 1$. Then the total premium is

$$E(S) + \sum_{i=1}^{n} \gamma_i f_i^{2p}\,\mathrm{Var}(S)^p\,.$$

We are looking for values of f_i for which this expression is minimal. To apply the method of Lagrange multipliers, we start with the function

$$G(f_1, f_2, \ldots, f_n, \lambda) = \sum_{i=1}^{n} \gamma_i f_i^{2p}\,\mathrm{Var}(S)^p - \lambda\left(\sum_{i=1}^{n} f_i - 1\right).$$

The partial derivatives are

$$\frac{\partial G}{\partial f_k} = 2p\gamma_k f_k^{2p-1}\,\mathrm{Var}(S)^p - \lambda f_k\,, \qquad k = 1, 2, \ldots, n\,.$$

If we set them equal to 0 and solve for f_k, we get

$$\tilde{f}_k = \left(\frac{\gamma}{\gamma_k}\right)^{\frac{1}{2p-1}}, \qquad k = 1, 2, \ldots, n\,,$$

where γ is a constant such that

$$\sum_{i=1}^{n} \tilde{f}_i = 1\,,$$

or equivalently,

$$\left(\frac{1}{\gamma}\right)^{\frac{1}{2p-1}} = \sum_{i=1}^{n} \left(\frac{1}{\gamma_i}\right)^{\frac{1}{2p-1}}.$$

The resulting total premium is

$$E(S) + \sum_{i=1}^{n} \gamma_i \tilde{f}_i^{2p}\,\mathrm{Var}(S)^p = E(S) + \gamma\,\mathrm{Var}(S)^p\,. \tag{1}$$

Note that this expression can be interpreted as the premium that results from applying a p-th power variance principle with parameters γ and p.
Two questions have to be answered. Firstly, the method of Lagrange multipliers gives necessary (but not always sufficient) conditions. How can we be sure that we have obtained the minimal premium? Secondly, is the solution also optimal in a global sense, i.e., if we do not limit the analysis to proportional decompositions of S?

4 Global optimality

To obtain a positive answer to these last two questions, we must show that

$$E(S) + \gamma\,\mathrm{Var}(S)^p \leq E(S) + \sum_{i=1}^{n} \gamma_i\,\mathrm{Var}(S_i)^p$$

for any random variables $S_1, S_2, \ldots, S_n$ such that $S = S_1 + S_2 + \cdots + S_n$. Since

$$\begin{aligned}\mathrm{Var}(S)^p &= \mathrm{Var}\left(\sum_{i=1}^{n} S_i\right)^p \\ &= \left(\sum_{i=1}^{n} \sigma_i^2 + 2\sum_{i=2}^{n}\sum_{j=1}^{i-1} \varrho_{ij}\sigma_i\sigma_j\right)^p \\ &\le \left(\sum_{i=1}^{n} \sigma_i\right)^{2p},\end{aligned}$$

where σ_i is the standard deviation of S_i and ϱ_{ij}, the correlation coefficient between S_i and S_j, it is sufficient to show that

$$\left(\sum_{i=1}^{n} \sigma_i\right)^{2p} \le \sum_{i=1}^{n} \frac{\gamma_i}{\gamma}\sigma_i^{2p}$$

for any positive numbers $\sigma_1, \sigma_2, \ldots, \sigma_n$. Let

$$q_i = \left(\frac{\gamma}{\gamma_i}\right)^{\frac{1}{2p-1}} \quad \text{and} \quad \beta_i = \left(\frac{\gamma_i}{\gamma}\right)^{\frac{1}{2p-1}}.$$

Then the last inequality can be written as

$$\left(\sum_{i=1}^{n} q_i\beta_i\sigma_i\right)^{2p} \le \sum_{i=1}^{n} q_i(\beta_i\sigma_i)^{2p}. \tag{2}$$

Note that $q_i > 0$ and $\sum_{i=1}^{n} q_i = 1$. So we can think of the q_i's as probabilities, and then by applying Jensen's inequality, we see that (2) is verified since $p > 0.5$. Therefore, (1) is indeed the minimum total premium in a global sense.

5 The limiting case $p = 0.5$

If all companies apply a standard deviation principle, the total premium is minimal if the risk is distributed among the companies with the smallest parameter value. This well known result can now be obtained as a limiting result from the p-th power variance principle. To fix ideas, suppose that $\gamma_1 < \gamma_i$, $i = 2, 3, \ldots, n$.

Then

$$
\begin{aligned}
\frac{1}{\gamma(p)} &= \left(\sum_{i=1}^{n}\left(\frac{1}{\gamma_i}\right)^{\frac{1}{2p-1}}\right)^{2p-1} \\
&= \frac{1}{\gamma_1}\left(1+\sum_{i=2}^{n}\left(\frac{\gamma_1}{\gamma_i}\right)^{\frac{1}{2p-1}}\right)^{2p-1} \\
&\rightarrow \frac{1}{\gamma_1} \qquad \text{for} \quad p \downarrow 0.5\,,
\end{aligned}
$$

which shows that $\tilde{f}_1(p) \rightarrow 1$ for $p \downarrow 0.5$. And since $\tilde{f}_i(p) > 0$ and $\sum_{i=1}^{n} \tilde{f}_i = 1$, it follows that $\tilde{f}_i(p) \rightarrow 0$ for $p \downarrow 0.5$, $i = 1, 2, \ldots, n$.

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Some results on a model in risk theory with constant dividend barrier

1. In 1974 H. Gerber [5] considered a new mathematical model in relation to actuarial strategies in presence of a linear dividend barrier; this model overcomes old polemics about payment of dividends and safety of an Insurance Company (cf. [3]).
In this framework, we suppose that the process of the aggregate claims $\{S_t\}$ is a compound Poisson process with parameter $\gamma > 0$; in the following we consider the case of purely positive risk sums, so, if S is the distribution function of the risk sums, the mean claim size is given by:

$$\mu_1 = \int_0^{+\infty} v\, dS(v). \tag{1}$$

Therefore, if C is the constant income, continuously received in time, the following condition holds:

$$C > \gamma\mu_1 .$$

According to Gerber's model (cf. also [6] page 139), we suppose that when the surplus reaches a fixed barrier, dividends are paid and the surplus waits on the barrier until the next claim.
In [5] and then in [7], the case of a linear dividend barrier b_t depending on time is considered:

$$b_t = b + at, \tag{2}$$

with $b \geq 0$, $0 < a \leq C$, so that the risk reserve X_t verifies the conditions:

$$\begin{aligned} dX_t &= C\,dt - dSt \qquad \text{if} \quad X_t < b + at \\ dX_t &= a\,dt - dSt \qquad \text{if} \quad X_t = b + at. \end{aligned} \tag{3}$$

If no barrier of this type is fixed, we set $a = C$.

2. We denote by $\psi(x, b)$ the ultimate ruin probability in presence of a linear dividend barrier, x being the initial risk reserve; considering all possible events, it is soon verified that (cf. [7]) the corresponding survival probability $U(x,b)$ satisfies the following equation:

$$C\frac{\partial U}{\partial x}+a\frac{\partial U}{\partial b}-\gamma U+\gamma\int_0^x U(x-y,b)\,dS(y)=0, \tag{4}$$

with the boundary condition:

$$\left(\frac{\partial U}{\partial x}\right)_{x=b}=0. \tag{5}$$

Denoting by $W(x,b)$ the expectation of the discounted dividend payments, Gerber shows (cf. [7] page 109) that $W(x,b)$ is the unique solution of the following integro-differential problem:

$$\begin{cases} -C\dfrac{dW}{dx}-a\dfrac{\partial W}{\partial b}+(\delta+\gamma)W=\gamma\displaystyle\int_0^x W(x-y,b)\,dS(y) \\ \left(\dfrac{\partial W}{\partial x}\right)_{x=b}=1 \qquad (0\le x\le b<+\infty) \end{cases} \tag{6}$$

δ being a constant force of interest at which the divided payments are discounted, and he gives the expression of $W(x,b)$, under the assumption that $S(v)=1-e^{-v}$. Gerber's model can be generalized (cf. [4], [8]) by supposing that the income depends from the initial surplus x by a given rate θ, so we consider the income c, received as premiums, and the interest at rate θ:

$$C(x)=c+\theta x. \tag{7}$$

In this scheme of ideas, we want to evaluate the expectation of the discounted dividend payments.

3. Under the assumption that the process of the aggregate claims is a compound Poisson one (with parameter γ), the mathematical model describing the surplus of an Insurance Company in presence of a linear dividend barrier generates the following integro-differential problem:

$$\begin{cases} -(c+\theta x)\dfrac{\partial W}{\partial W}-a\dfrac{\partial W}{\partial b}+(\delta+\gamma)W=\gamma\displaystyle\int_0^x W(x-y,b)\,dS(y) \\ \left(\dfrac{\partial W}{\partial x}\right)_{x=b}=1 \end{cases} \tag{8}$$

where $c(>0)$ is the premium density, $\theta(>0)$ the interest rate on the initial surplus, $\delta(>0)$ the force of interest at which the dividend payments are discounted.
The unique solution of (8) is the expectation of the discounted dividend payments.
In the case of a constant barrier, i.e. $a = 0$, we can treat b as a fixed parameter and consider the function $W(.,b)$ as a function of the only x variable, then it follows from (8) that $W(.,b)$ is the unique solution of the integro-differential equation:

$$-(c+\theta x)W'(x,b) + (\delta+\gamma)W(x,b) = \gamma \int_0^x W(x-y,b)\, dS(y) \qquad (9)$$

combined with the boundary condition:

$$W'(b,b) = 1. \qquad (10)$$

In order to compute $W(.,b)$, we consider the equation:

$$-(c+\theta x)h'(x) + (\delta+\gamma)h(x) = \gamma \int_0^x h(x-y)\, dS(y), \qquad (11)$$

which has a unique solution, apart from a multiplicative constant. Let h be a solution of (11), then it is clear that $W(.,b)$ and h are linked by the following relation:

$$W(x,b) = \frac{h(x)}{h'(b)}, \qquad (0 \le x \le b < +\infty), \qquad (12)$$

therefore we only have to find a solution h of (11). We now consider the special case where the claim amounts are exponentially distributed, i.e. $S(y) = 1 - e^{-y}$. In this case the equation (11) can be written as follows:

$$-(c+\theta x)h'(x) + (\delta+\gamma)h(x) = \gamma \int_0^x h(y)e^{-(x-y)}\, dy. \qquad (13)$$

It is clear that the solutions of (13) are at least of class C^∞, hence by multiplying the equation (13) with e^x and differentiating with respect to x, we obtain:

$$-(c+\theta x)\frac{d^2h}{dx^2} + [\delta+\gamma-\theta-(c+\theta x)]\frac{dh}{dx} + \delta h = 0. \qquad (14)$$

The equation (14) can be studied by means of classical techniques of the theory of hypergeometric equations (cf. [9]). We define:

$$w(y) = h\left(y - \frac{c}{\theta}\right)e^{y}, \qquad y = x + \frac{c}{\theta}, \tag{15}$$

since

$$\frac{dh}{dx} = e^{-y}\frac{dw}{dy} - e^{-y}w,$$
$$\frac{d^2h}{dx^2} = \frac{d^2w}{dx^2}e^{-y} - 2\left(e^{-y}\frac{dw}{dy} - e^{-y}w\right) - e^{-y}w,$$

and substituting in (14), we get:

$$y\frac{d^2w}{dy^2} + \left(1 - \frac{\delta+\gamma}{\theta} - y\right)\frac{dw}{dy} - \left(1 - \frac{\gamma}{\theta}\right)w = 0, \qquad y \geq 0. \tag{16}$$

(16) is an hypergeometric confluent equation for w.

On the basis of the above considerations, it follows that if h is a solution of equation (13) in $\left(-\frac{c}{\theta}, +\infty\right)$, then

$$h(x) = e^{-x-c/\theta}w\left(x + \frac{c}{\theta}\right), \tag{17}$$

where w is a solution of (16) in $(0, +\infty)$.

If $[\theta - (\delta+\gamma)]/\theta$ is not an integer, the general solution of equation (16) is given by:

$$w(y) = A\Phi\left(1 - \frac{\gamma}{\theta}, \frac{\theta - (\delta+\gamma)}{\theta}; y\right)$$
$$+ By^{(\delta+\gamma)/\theta}\Phi\left(1 + \frac{\delta}{\theta}, 1 + \frac{\delta+\gamma}{\theta}; y\right), \tag{18}$$

A and B being two arbitrary constants and $\Phi(\alpha, \beta; z)$ denoting the Kummer's function:

$$\Phi(\alpha, \beta; z) = 1 + \sum_{k=1}^{\infty} \frac{\alpha(\alpha+1)\ldots(\alpha+k-1)}{\beta(\beta+1)\ldots(\beta+k-1)}\frac{z^k}{k!}.$$

By virtue of (17), the solution $h(x)$ of equation (13) can be expressed by the formula:

$$h(x) = e^{-x-c/\theta} A\Phi\left(1 - \frac{\gamma}{\theta}, \frac{\theta - (\delta+\gamma)}{\theta}; x + \frac{c}{\theta}\right)$$
$$+ e^{-x-c/\theta} B\left(x + \frac{c}{\theta}\right)^{(\delta+\gamma)/\theta} \Phi\left(1 + \frac{\delta}{\theta}, 1 + \frac{\delta+\gamma}{\theta}; x + \frac{c}{\theta}\right). \quad (19)$$

We now observe that (cf. [1]):

$$e^{-z}\Phi(\alpha, \beta; z) = \Phi(\beta - \alpha, \beta; -z), \quad (20)$$

and that the condition $[\theta - (\delta + \gamma)]/\theta \notin \mathbb{Z}$ is equivalent to $(\delta + \gamma)/\theta \notin \mathbb{N}$ $(\delta, \gamma, \theta > 0)$, therefore under the condition $(\delta + \gamma)/\theta \notin \mathbb{N}$, we have from (19):

$$h(x) = A\Phi\left(-\frac{\delta}{\theta}, 1 - \frac{\delta+\gamma}{\theta}; -x - \frac{c}{\theta}\right)$$
$$+ B\left(x + \frac{c}{\theta}\right)^{(\delta+\gamma)/\theta} \Phi\left(\frac{\gamma}{\theta}, 1 + \frac{\delta+\gamma}{\theta}; -x - \frac{c}{\theta}\right). \quad (21)$$

In order to determine a relation between A and B, we replace the expression of $h(x)$ given in (21) into (13), obtaining for $x = 0$:

$$-c\frac{\partial}{\partial x}\left[A\Phi\left(-\frac{\delta}{\theta}, 1 - \frac{\delta+\gamma}{\theta}; -x - \frac{c}{\theta}\right)\right]_{x=0}$$
$$+ (\delta+\gamma)A\Phi\left(-\frac{\delta}{\theta}, 1 - \frac{\delta+\gamma}{\theta}; -\frac{c}{\theta}\right)$$
$$-c\frac{\partial}{\partial x}\left[B\left(x + \frac{c}{\theta}\right)^{(\delta+\gamma)/\theta} \Phi\left(\frac{\gamma}{\theta}, 1 + \frac{\delta+\gamma}{\theta}; -x - \frac{c}{\theta}\right)\right]_{x=0}$$
$$+ (\delta+\gamma)B\left(\frac{c}{\theta}\right)^{(\delta+\gamma)/\theta} \Phi\left(\frac{\gamma}{\theta}, 1 + \frac{\delta+\gamma}{\theta}; -\frac{c}{\theta}\right) = 0. \quad (22)$$

Formula (22) gives the required condition for A and B.

The function Φ satisfies:

$$\frac{d}{dz}\Phi(\alpha, \beta; z) = \frac{\alpha}{\beta}\Phi(\alpha + 1, \beta + 1; z).$$

From (22) we have immediately:

$$A\left[-\frac{c\delta}{\theta-\delta-\gamma}\Phi\left(1-\frac{\delta}{\theta},2-\frac{\delta+\gamma}{\theta};-\frac{c}{\theta}\right)\right.$$
$$\left.+(\delta+\gamma)\Phi\left(-\frac{\delta}{\theta},1-\frac{\delta+\gamma}{\theta};-\frac{c}{\theta}\right)\right]$$
$$+B\left[-c\left(\frac{c}{\theta}\right)^{(\delta+\gamma)/\theta}\frac{\gamma}{\theta+\delta+\gamma}\Phi\left(1+\frac{\gamma}{\theta},2+\frac{\delta+\gamma}{\theta};-\frac{c}{\theta}\right)\right]=0, \quad (23)$$

hence:

$$B=A\,K(c,\delta,\gamma,\theta),$$

with

$$K(c,\delta,\gamma,\theta)=-\frac{\frac{c\delta}{\theta-\delta-\gamma}\Phi\left(1-\frac{\delta}{\theta},2-\frac{\delta+\gamma}{\theta};-\frac{c}{\theta}\right)}{c\left(\frac{c}{\theta}\right)^{(\delta+\gamma)/\theta}\frac{\gamma}{\theta+\delta+\gamma}\Phi\left(1+\frac{\gamma}{\theta},2+\frac{\delta+\gamma}{\theta};-\frac{c}{\theta}\right)}$$
$$-\frac{(\delta+\gamma)\Phi\left(-\frac{\delta}{\theta},1-\frac{\delta+\gamma}{\theta};-\frac{c}{\theta}\right)}{c\left(\frac{c}{\theta}\right)^{(\delta+\gamma)/\theta}\frac{\gamma}{\theta+\delta+\gamma}\Phi\left(1+\frac{\gamma}{\theta},2+\frac{\delta+\gamma}{\theta};-\frac{c}{\theta}\right)}. \quad (24)$$

We get from (21):

$$h(x)=A\left[\Phi\left(-\frac{\delta}{\theta},1-\frac{\delta+\gamma}{\theta};-x-\frac{c}{\theta}\right)\right.$$
$$\left.+K(c,\delta,\gamma,\theta)\left(x+\frac{c}{\theta}\right)^{(\delta+\gamma)/\theta}\Phi\left(\frac{\gamma}{\theta},1+\frac{\delta+\gamma}{\theta};-x-\frac{c}{\theta}\right)\right]$$

and being:

$$\frac{dh}{dx}(b)=A\left[\frac{\delta}{\theta-\delta-\gamma}\Phi\left(1-\frac{\delta}{\theta},2-\frac{\delta+\gamma}{\theta};-b-\frac{c}{\theta}\right)\right.$$
$$+K(c,\delta,\gamma,\theta)\frac{\delta+\gamma}{\theta}\left(b+\frac{c}{\theta}\right)^{-1+(\delta+\gamma)/\theta}$$
$$\times\Phi\left(\frac{\gamma}{\theta},1+\frac{\delta+\gamma}{\theta};-b-\frac{c}{\theta}\right)$$
$$-K(c,\delta,\gamma,\theta)\left(b+\frac{c}{\theta}\right)^{(\delta+\gamma)/\theta}$$
$$\left.\times\frac{\gamma}{\theta+\delta+\gamma}\Phi\left(\frac{\gamma}{\theta}+1,\,2+\frac{\delta+\gamma}{\theta};-b-\frac{c}{\theta}\right)\right], \quad (25)$$

finally, by virtue of (12), we can deduce the following result:

Theorem 3.1. Let $W(x,b)$ be the expectation of the discounted dividend payments in the presence of a constant dividend barrier b, x being the initial surplus. Suppose that the aggregate claims process is compound Poisson (with Poisson parameter γ and exponentially distributed claim amounts). Let c denote the premium density, θ the interest rate on the initial surplus, δ the force of interest at which the dividend payments are discounted. $W(x,b)$ satisfies the equation (9). If $\frac{\delta+\gamma}{\theta} \notin \mathbb{N}$ when $W(x,b)$ is given by:

$$\begin{aligned} W(x,b) = \Bigg[& \Phi\left(-\frac{\delta}{\theta}, 1-\frac{\delta+\gamma}{\theta}; -x-\frac{c}{\theta}\right) \\ & + K(c,\delta,\gamma,\theta)\left(x+\frac{c}{\theta}\right)^{(\delta+\gamma)/\theta} \Phi\left(\frac{\gamma}{\theta}, 1+\frac{\delta+\gamma}{\theta}; -x-\frac{c}{\theta}\right)\Bigg] \\ & \times \Bigg[\frac{\delta}{\theta-\delta-\gamma}\Phi\left(1-\frac{\delta}{\theta}, 2-\frac{\delta+\gamma}{\theta}; -b-\frac{c}{\theta}\right) \\ & + K(c,\delta,\gamma,\theta)\frac{\delta+\gamma}{\theta}\left(b+\frac{c}{\theta}\right)^{-1+(\delta+\gamma)/\theta} \\ & \times \Phi\left(\frac{\gamma}{\theta}, 1+\frac{\delta+\gamma}{\theta}; -b-\frac{c}{\theta}\right) \\ & - K(c,\delta,\gamma,\theta)\left(b+\frac{c}{\theta}\right)^{(\delta+\gamma)/\theta} \\ & \times \frac{\gamma}{\theta+\delta+\gamma}\Phi\left(1+\frac{\gamma}{\theta}, 2+\frac{\delta+\gamma}{\theta}; -b-\frac{c}{\theta}\right)\Bigg]^{-1}, \end{aligned}$$

with $K(c,\delta,\gamma,\theta)$ given by (24).

4. According to the scheme of the previous sections, we now want to evaluate the expectation of the discounted dividend payments when the process of the aggregate claims is a Wiener one. It is well known (cf. [2]) that processes of this type naturally appear in models in which the insurance companies have a large number of customers.

The presence of a linear dividend barrier implies that the risk reserve X_t verifies the condition:

$$dX_t = \mu\, dt + dW_t, \qquad \text{if} \quad X_t < b + at, \tag{26}$$

$\mu(>0)$ denoting the constant income and $\{W_t\}$ the standard Wiener process. It is known (cf. [7] page 112) that in this case the probability of survival $U(x,b)$ satisfies the following partial differential equation:

$$\frac{1}{2}\frac{\partial^2 U}{\partial x^2}+\mu\frac{\partial U}{\partial x}+a\frac{\partial U}{\partial b}=0,$$

together with the conditions:

$$U(0,b)=0,$$
$$\left(\frac{\partial U}{\partial x}\right)_{x=b}=0,$$
$$\lim_{b\to+\infty} U(x,b)=1-e^{-2\mu x}.$$

Let us consider in this model the expectation of the discounted dividend payments $W(x,b)$. If $a=0$, we can treat b as a fixed parameter; then denoting by δ the force of interest at which the dividend payments are discounted, the function $W(.,b)$, considered as a function of the only x variable, satisfies the differential equation (cf. [7]):

$$\frac{1}{2}W''(x,b)+\mu W'(x,b)-\delta W(x,b)=0, \tag{27}$$

with the conditions:

$$W(0,b)=0, \tag{28}$$
$$W'(b,b)=1. \tag{29}$$

Introducing interest in the above described model (θ denoting as in the previous section the interest rate on the initial surplus), we are lead to study the following equation:

$$\frac{1}{2}W''(x,b)+(\mu+\theta x)W'(x,b)-\delta W(x,b)=0, \tag{30}$$

together with the conditions (28) and (29).
It is well known that every solution of (30) with the above boundary conditions can be expressed as the sum of a series of the type $\sum_{n=1}^{\infty} c_n x^n$; therefore if we set:

$$W(x,b)=\sum_{n=1}^{\infty} c_n x^n, \tag{31}$$

it immediately follows that:

$$W'(x,b) = \sum_{n=1}^{\infty} n c_n x^{n-1}, \tag{32}$$

$$W''(x,b) = \sum_{n=1}^{\infty} n(n-1) c_n x^{n-2}. \tag{33}$$

By replacing (31), (32) and (33) into (30), we obtain:

$$c_2 = -\mu c_1 \,, \tag{34}$$

and

$$\frac{(n+1)(n+2)}{2} c_{n+2} = -\mu(n+1)c_{n+1} + (\delta - \theta n)c_n \,, \quad n \geq 1$$

which is equivalent to:

$$c_{n+2} = -\frac{2\mu}{(n+2)} c_{n+1} + \frac{2(\delta - \theta n)}{(n+1)(n+2)} c_n, \qquad n \geq 1. \tag{35}$$

Remembering that $c_1 = W'(0,b)$, we get by (34) $c_2 = -\mu W'(0,b)$.
It follows by (35) that c_n are proportional to $W'(0,b)$. We then have:

$$c_n = W'(0,b)\gamma_n(\mu,\delta,\theta), \qquad n \geq 1$$

$\gamma_1(\mu,\delta,\theta)$ being equal to 1, and $\gamma_2(\mu,\delta,\theta)$ being equal to $-\mu$.
The coefficients γ_n follow the same recursion formula as the coefficients c_n: i.e.

$$\gamma_{n+2} = -\frac{2\mu}{(n+2)} \gamma_{n+1} + \frac{2(\delta - \theta n)}{(n+1)(n+2)} \gamma_n \,, \qquad n \geq 1. \tag{36}$$

Taking into account the boundary condition (29) we actually find the following result:

Theorem 4.1. Let $W(x,b)$ be the expectation of the discounted dividend payments in the presence of a constant dividend barrier b, x being the initial surplus. Suppose that the process of the aggregate claims is standard Wiener. Let μ denote the premium density, θ the interest rate on the initial surplus, δ the force of interest at which the dividend payments are discounted.
$W(x,b)$ satisfies the equation (30).

Then $W(x, b)$ is given by:

$$W(x,b) = \frac{\sum\limits_{n=1}^{\infty} \gamma_n x^n}{\sum\limits_{n=1}^{\infty} n\gamma_n b^{n-1}},$$

with $\gamma_1 = 1$, $\gamma_2 = -\mu$ and

$$\gamma_n = -\frac{2\mu}{n}\gamma_{n-1} + \frac{2(\delta - \theta(n-2))}{(n-1)n}\gamma_{n-2}, \qquad n \geq 3.$$

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An Elementary Proof of the Schuette-Nesbitt Formula

Let $A_1, \ldots, A_m \subset \Omega$ be m events, denote by $P_{[j]}$ the probability that exactly j of the m events take place and by $P_{i_1 \ldots i_k}$ the probability that the specified k events $A_{i_1}, \ldots, A_{i_k}$ occur, irrespective of the occurrence of the other $m-k$ events. The Schuette-Nesbitt formula, a central tool in multiple life theory, states that, for any real numbers $c_0, \ldots, c_m$,

$$\sum_{j=0}^{m} c_j P_{[j]} = \sum_{k=0}^{m} \Delta^k c_0 \sum_{1 \le i_1 < \cdots < i_k \le m} P_{i_1 \ldots i_k} \, .$$

Here the difference operator Δ is defined by $\Delta c_\ell = c_{\ell+1} - c_\ell$, i.e.

$$\Delta^k c_0 = \sum_{\ell=0}^{k} (-1)^{k+\ell} \binom{k}{\ell} c_\ell \, .$$

The proof in standard textbooks (cf. [1] or [2]) is based on an elegant manipulation of the difference operator Δ and the shift operator E defined by $Ec_\ell = c_{\ell+1}$. The main step consists in calculating expectations of certain functions of these operators. Here we present an elementary proof, which also shows that the Schuette-Nesbitt formula is rather a combinatorial theorem than a probabilistic one.

Assume that, for $j = 0, \ldots, m$, to each set which is the intersection of exactly j sets $A_{r_1}, \ldots, A_{r_j}$ and $m-j$ sets $A^c_{s_1}, \ldots, A^c_{s_{m-j}}$ (where $\{s_1, \ldots, s_{m-j}\} = \{1, \ldots, m\} \setminus \{r_1, \ldots, r_j\}$ and $A^c_s = \Omega \setminus A_s$) the same weight c_j is to be assigned by assigning to the set Ω a weight w_0 and, for $k = 1, \ldots, m$, to each intersection of exactly k sets $A_{i_1}, \ldots, A_{i_k}$ the same weight w_k. (A weight of a set is a not necessarily nonnegative real number by which the measure of the set ist multiplied.) If the weights w_k are given, the weight c_j of any set $A_{r_1} \cap \ldots \cap A_{r_j} \cap A^c_{s_1} \cap \ldots \cap A^c_{s_{m-j}}$ is the sum of the weight w_0 of the set Ω, the weights w_1 of each of the j sets $A_{r_1}, \cdots, A_{r_j}$, the weights w_2 of each of the $\binom{j}{2}$ sets $A_{r_1} \cap A_{r_2}, \ldots, A_{r_{j-1}} \cap A_{r_j}$, and, generally, the weights w_k of each of the $\binom{j}{k}$ sets $A_{i_1} \cap \ldots \cap A_{i_k}$ which are obtained by choosing k numbers $i_1, \ldots, i_k$ out of the j numbers $r_1, \ldots, r_j$. Thus $c_j = \sum_{k=0}^{j} \binom{j}{k} w_k$.

Consequently (as the inverse of the matrix $\left(\binom{j}{k}\right)_{j=0,\ldots,m;\,k=0,\ldots,m}$ is the matrix $\left((-1)^{k+\ell}\binom{k}{\ell}\right)_{k=0,\ldots,m;\,\ell=0,\ldots,m}$), the desired weights $c_0,\ldots,c_m$ arise if and only if, for $k=0,\ldots,m$, $w_k=\sum_{\ell=0}^{k}(-1)^{k+\ell}\binom{k}{\ell}c_\ell=\Delta^k c_0$.

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Clarification of certain extensions of the chain-ladder technique

After reading further papers on nonlinear time series models (e.g. Pemberton (1987)), the author noticed that his theorem 1 in Kremer (1993) cannot be true in general. One can only say that:

$$E(X_{ij} \mid X_{lk},\, l + k < i + j) = f_{j,a_j}(X_{i,j-1}),$$

what also suggests to forecast the $X_{i,n-i+k'}$ for $k \geq 3$, according to (3.6). But the resulting forecasts are not the optimal ones of (3.5) in general. They are something like handy, reasonable approximations to the optimal ones. It does not seem to be possible to give very handy, analytical expressions for the forecast defined by (3.5) in the general nonaffine case. Consequently one will be satisfied in practice by applying (3.6).

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Die Sterblichkeit in der Schweiz in jüngster Zeit

In der vorliegenden Studie wird anhand der Sterbefälle die Sterblichkeit in der Schweiz von 1990/1991 mit derjenigen der schweizerischen Sterbetafel 1978/1983 verglichen. Das Vorgehen sei am folgenden Beispiel, die männliche Bevölkerung der Schweiz betreffend, erläutert: Laut Angaben in den Statistischen Jahrbüchern der Schweiz starben im Jahre 1990 32 492, im Jahre 1991 32 076 männliche Einwohner in unserem Lande. Das arithmetische Mittel aus beiden Zahlen, 32 284, ergibt die Zahl der in der Schweiz vom 1. Juli 1990 bis zum 30. Juni 1991 gestorbenen Männer. Nun stellt sich die Frage: Wieviele Männer wären in der Schweiz im Laufe derselben Zeitspanne gestorben, wenn ihr Ableben gemäss der Sterbetafel SM 1978/1983 erfolgt wäre? Auf der Basis des von der Volkszählung vom 4. Dezember 1990 ermittelten Bestandes der männlichen Bevölkerung der Schweiz errechnet sich diese Zahl wie folgt: Laut dieser Volkszählung betrug die Zahl der nulljährigen Knaben 39 354. Nach der Tafel SM 1978/1983 ist die Sterbewahrscheinlichkeit im Alter 0 gleich 0.009 487. Somit wären $0.009\,487 \times 39\,354 = 373$ Knaben im ersten Lebensjahr gestorben. Eine analoge Rechnung ergäbe 37 Tote unter den einjährigen Knaben, usw. Man errechnet für den Gesamtbestand 35 634 Sterbefälle. Diese Zahl, die, wie gesagt, unter der Voraussetzung (Supposition) ermittelt wurde, die Sterblichkeit von 1990/1991 verlaufe gemäss der Sterbetafel SM 1978/1983, wird nachstehend *supponierte* Sterblichkeit genannt, im Gegensatz zur effektiven Sterblichkeit. Im vorliegenden Beispiel liegt die effektive Sterblichkeit unter der supponierten; die Differenz beträgt 3 350 Todesfälle oder 9.4 % der supponierten Sterblichkeit.

In der folgenden Tabelle I sind für die schweizerische Wohnbevölkerung von 1990/1991 die Rechenergebnisse – getrennt nach Geschlecht und aufgeteilt in Altersgruppen – zusammengestellt. (Es versteht sich von selbst, dass für Frauen die supponierte Sterblichkeit aufgrund der Tafel SF 1978/1983 ermittelt wurde).

Tabelle I: Sterbefälle in der schweizerischen Bevölkerung von 1990/1991 (Alle Todesursachen)

Sterbefälle der Männer						Sterbefälle der Frauen				
0–14	15–29	30–64	65+	Total	Altersgruppen	0–14	15–29	30–64	65+	Total
614	1 129	8 402	25 489	35 634	supponiert	415	385	4 258	30 322	35 380
477	1 094	7 191	23 522	32 284	effektiv	352	338	3 605	26 606	30 901
137	35	1 211	1 967	3 350	Differenz	63	47	653	3 716	4 479

Was bereits für die männliche Bevölkerung der Schweiz festgestellt wurde, gilt, wie Tabelle I deutlich zeigt, für beide Geschlechter in allen Altersgruppen: Die effektive Sterblichkeit von 1990/1991 liegt unter der aufgrund der Sterbetafel 1978/1983 berechneten supponierten Sterblichkeit, was auf einen günstigen Sterblichkeitsverlauf verweist.
Über die Zahl der supponierten und effektiven Sterbefälle infolge Krebs- und Kreislaufkrankheiten sowie gewaltsamen Tod, den drei wichtigsten Todesursachen, orientiert Tabelle II. Der Rechenprozess ist gleich wie in Tabelle I. Die supponierten Sterbefälle wurden aufgrund der einjährigen abhängigen Sterbewahrscheinlichkeiten ermittelt.

Tabelle II: Sterbefälle in der schweizerischen Wohnbevölkerung 1990/1991 infolge von Krebskrankheiten, Kreislaufkrankheiten und gewaltsamem Tod

Sterbefälle der Männer						Sterbefälle der Frauen				
0–14	15–29	30–64	65+	Total	Altersgruppen	0–14	15–29	30–64	65+	Total
					Krebskrankheiten					
34	86	2 712	6 393	9 225	supponiert	28	54	2 002	5 523	7 607
25	63	2 396	6 798	9 282	effektiv	21	36	1 816	5 685	7 558
9	23	316	–405	–57	Differenz	7	18	186	–162	49
					Kreislaufkrankheiten					
7	27	2 414	10 284	12 732	supponiert	4	18	707	13 501	14 230
10	49	1 967	10 876	12 902	effektiv	7	20	630	14 152	14 809
–3	–22	447	–592	–170	Differenz	–3	–2	77	–651	–579

Tabelle II: Fortsetzung

Sterbefälle der Männer						Sterbefälle der Frauen				
0–14	15–29	30–64	65+	Total	Altersgruppen	0–14	15–29	30–64	65+	Total
					Gewaltsamer Tod					
126	892	1 500	1 041	3 559	supponiert	69	239	523	1 295	2 126
83	802	1 334	1 076	3 295	effektiv	45	184	483	1 349	2 061
43	90	166	–35	264	Differenz	24	55	40	–54	65

Tabelle II bietet ein anderes Bild als Tabelle I. So weicht die effektive Zahl der Krebstoten nur wenig von der supponierten Zahl ab; somit hat sich die Wahrscheinlichkeit, an Krebs zu sterben, wenig geändert. Dies gilt auch für den gewaltsamen Tod bei Frauen. Bei den Kreislaufkrankheiten liegt die Zahl der effektiven Sterbefälle in fast allen Altersgruppen deutlich über der supponierten. Summiert man in der Kolonne Total der Tabelle II die Zahl aller supponierten sowie die Zahl aller effektiven Sterbefälle,so erhält man:

Supponierte Sterbefälle	49 479
Effektive Sterbefälle	49 907
Differenz	–428

Die Zahl der effektiven Sterbefälle infolge der drei wichtigsten Todesursachen übertrifft somit die entsprechende supponierte Zahl. Daraus folgt, dass der anhand der Tabelle I nachgewiesene günstige Sterblichkeitsverlauf nicht auf diese drei Todesursachen, sondern auf die übrigen Todesursachen (wie Infektionskrankheiten, Erkrankung der Atmungsorgane, etc.) zurückzuführen ist.
Bildet man die Differenzen zwischen den entsprechenden Werten der Tabellen I und II, so erhält man Tabelle III, die die Zahl der Sterbefälle infolge dieser übrigen Todesursachen angibt:

Tabelle III: Sterbefälle in der schweizerischen Wohnbevölkerung 1990/1991 infolge der übrigen Todesursachen

Sterbefälle der Männer						Sterbefälle der Frauen				
0–14	15–29	30–64	65+	Total	Altersgruppen	0–14	15–29	30–64	65+	Total
447	124	1 776	7 771	10 118	supponiert	314	74	1 026	10 003	11 417
359	180	1 494	4 772	6 805	effektiv	279	98	676	5 420	6 473
88	–56	282	2 999	3 313	Differenz	35	–24	350	4 583	4 944

In der Kolonne Total der Tabelle III beträgt die Differenz zwischen den supponierten und effektiven Sterbefällen für Männer 32,7 %, für Frauen 43.3 % der supponierten Sterbefälle. Diese grossen Prozentsätze bestätigen die These, dass der günstige Sterblichkeitsverlauf von 1990/1991 auf diese übrigen Todesursachen zurückzuführen ist.

Anmerkung: Die den Berechnungen zugrunde liegenden Sterbewahrscheinlichkeiten sind der folgenden Publikation des Bundesamtes für Statistik entnommen:
Amtliche Statistik der Schweiz, Nr. 150 Schweizerische Sterbetafel 1978/1983, Bern 1988.

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