Werner Hürlimann, Winterthur

Transforming, ordering and rating risks

Introduction

Due to the seminal work by Rothschild and Stiglitz (1970/71/72), the theory of stochastic orderings and their applications has seen a growing interest and a tremendous development during the last two decades. The main results obtained so far in this area have been collected by Shaked and Shanthikumar (1994). On the other side Mosler and Scarsini (1993) have compiled an extensive classified bibliography.
In actuarial science the corresponding development starts with the landmark Bühlmann et al. (1977) and owes much to Goovaerts et al. (1984/90), the two thesis by Van Heerwaarden (1991) and Kling (1993), and the new monograph by Kaas et al. (1994). The subject is now established and the potential for new questions and applications has not yet been fully explored. The present paper deals with examples of "risk measures", which are studied from an axiomatic point of view in Ramsay (1993). As observed by Garrido (1993), p. 340, two different concepts must be distinguished. The "absolute" risk measure takes into account the "variability" of a risk, or the "level of risk" inherent in it, and, once defined (which is an open problem), the loading of an insurance premium is a function of this quantity. The second "relative" risk measure considers riskiness relative to another reference scale. For example the loss ratio risk leads to the expected loss ratio as relative risk measure. Our search of a sound "risk measure" points therefore towards two directions. In a first part (Sections 1 to 4) new properties and actuarial applications of a stable retention ratio and stable return index, "relative" risk measures introduced previously in Hürlimann (1992), are presented. The second part (Section 5 and 6) contains a further analysis of two "absolute" risk measures recently considered by the author. Though no direct connections between the two parts are given, they both are strongly based on unifying ordering properties of probabilistic real functionals (or transformed random variables), namely preservation of stochastic dominance and stop-loss order. A more detailed outline follows.
In Section 1 the definition of *loss ratio ordering* is recalled and shown to be a generalization of the k-ordering by Heilmann (1985/86), which

refers in particular to Fischburn (1984). In Section 2 known properties of two "relative" risk measures, derived from the loss ratio ordering, are reviewed. Then in Section 3 the loss ratio orderings are interpreted in terms of stochastic dominance relations between transformed risks using the method by Heilmann (1986). The main part of actuarial interest in "relative" risk measures is found in Section 4, where classical applications in non-life (rating large claims in (re)insurance) and financial option pricing are presented. Example 4.1 shows the existence of infinitely many arbitrarily stop-loss ordered Pareto risks, which are all equal in total return and stable return order. For a special Dutch premium principle the non-existence of loss ratio ordered risks with equal mean of risks is settled in Proposition 4.1. It is shown in Example 4.3 that stable retention ratios and prices for lognormal risks with equal mean are preserved under stop-loss order. This provides an elementary proof of the fact that a rate of return on financial assets (by constant risk-free rate) has to increase by increasing volatility in the classical Black-Scholes option pricing model. Then in Example 4.4 Pareto risks with equal mean, which are less dangerous in stop-loss order and pricing order, are shown to possess higher stable return indices and stable retention ratios.

The problem of defining an adequate "absolute" risk measure is closely related to the theory of premium calculation principles, and no ultimate answer has been found so far. In this respect Section 5 reports of a (relative) good news, which finds by the way a further interesting financial application in "portfolio insurance" (see Section 6 in Hürlimann (1995a)). As shown in Example 6.1, the merit of the generalized Dutch premium principle must be appreciated with care: the premiums of long tailed risks can eventually not be discriminated from the premiums of other risks using this pricing principle. In our opinion the validity of a sound "absolute" risk measure or/and premium calculation principle must besides desirable properties specify the necessarily restricted space of risks on which it should be applied. As an example a second degree stop-loss order preserving splitting premium principle for an insurance economic environment of (positive) risks with fixed mean, variance and net stop-loss premium to a given deductible, is constructed. This solves a question suggested to us by Briys (1990), p. 37.

1 A loss ratio ordering based on stop-loss dominance

In the present paper X is a random variable, which represents a *risk* from life or non-life insurance, or from finance. To a risk X one associates a *risk price* $H[X]$ using a *pricing calculation principle* $H[\cdot]$. The use of such pricing principles in insurance risk theory, where they are called *premium calculation principles*, is well-known. In a finance context their use is more recent. Besides the risk itself we will consider the *loss ratio* transformed random variable $LH(X) = X/H[X]$ and the *return ratio* transformed random variable $GH[X] = 1 - LH(X)$.
Given loss ratio transforms $LH(X)$, $LH(Y)$ of risks X, Y, it is possible to define a *loss ratio ordering*, denoted $<_{LH}$, by the relation

$$X <_{LH} Y \quad \Leftrightarrow \quad LH(X) = \frac{X}{H[X]} <_{sl} LH(Y) = \frac{Y}{H[Y]}, \tag{1.1}$$

where $<_{sl}$ denotes the *stop-loss order* relation. Loss ratio ordering, or shortly LH-ordering, has been previously introduced by the author (1992) and is in fact a generalization of the *k-ordering* by Heilmann (1985), Section 4, and Heilmann (1986), Section 3.2, obtained in the special case of the mean value principle $H[X] = E[X]$:

$$\begin{aligned} X <_k Y \quad &\Leftrightarrow \quad X <_{LE} Y \\ &\Leftrightarrow \quad LE(X) = \frac{X}{E[X]} <_{sl} LE(Y) = \frac{Y}{E[Y]}. \end{aligned} \tag{1.2}$$

The k-ordering preserves the total order based on the comparison of the coefficients of variation of different risks:

$$X <_k Y \quad \Rightarrow \quad k[X] = \frac{\sqrt{\mathrm{Var}[X]}}{E[X]} \leq k[Y] = \frac{\sqrt{\mathrm{Var}[Y]}}{E[Y]}. \tag{1.3}$$

Therefore it preserves also the total premium order based on the comparison of "Karlsruhe" premiums $K[X] = (1 + k[X]^2)E[X]$ for risks with equal means. The significance of the latter premiums in *all-finance risk theory* (= actuarial and financial risk theory) has been studied by the author (1994a/95a/95b). "Karlsruhe" premiums can be derived from the "loss principle" of Heilmann (1988).
Pricing calculation principles should in principle satisfy the *mean exceeding* property $H[X] > E[X]$ (e.g. non-ruin or risk-profit arguments). For the

purpose of comparing loss ratios, it seems that the natural LH-ordering generalization of the k-ordering is more promising.
However all its useful properties have not yet been clarified. Furthermore its potential applications have not yet been sufficiently demonstrated.

2 Review of known properties of *LH*-ordering

In Hürlimann (1992) the LH-ordering has been introduced in connection with two "relative" risk measures, which should be useful when comparing loss ratios. The first one is the *H-stable retention ratio*, which can be obtained as fixed point $BH = BH[X]$ of the following "stop-loss" equation

$$BH = \pi_{LH(X)}(1 - BH), \tag{2.1}$$

with $\pi_{LH(X)}(d) = E[(LH(X) - d)_+]$, $d \in \mathbb{R}$, the stop-loss transform of the loss ratio transform. Its computation requires usually the knowledge of the probability distribution of the risk. By incomplete information, for example when only mean and variance are available, the simpler *H-stable return index*, which as a consequence of the inequality of Bowers (1969), is calculated as follows:

$$\delta H[X] = \frac{\mathrm{Var}[GH(X)]}{E[GH(X)]} = \frac{1}{H[X]} \cdot \frac{\mathrm{Var}[X]}{(H[X] - E[X])} . \tag{2.2}$$

This index measures per unit of risk price the relative variability of the risk with respect to the risk profit loading. The LH-ordering preserves the total order induced by stable retention ratios, called total *LH-return order* (author (1992), Theorem 1):

$$X <_{LH} Y \quad \Rightarrow \quad BH[X] \le BH[Y] . \tag{2.3}$$

For this reason it may be viewed as a stochastic generalization of the obvious total order concept of comparing stable retention ratios. On the other side the total order induced by the stable return index, called total *stable LH-return order*, is only preserved under the stronger loss ratio ordering with equal means, denoted by $<_{LH,=}$, which is defined by the relations

$$X <_{LH,=} Y \quad \Leftrightarrow \quad LH(X) <_{sl,=} LH(Y)$$

$$\Leftrightarrow \quad LH(X) <_{sl} LH(Y) \quad \text{and} \quad E[LH(X)] = E[LH(Y)] \tag{2.4}$$

This follows from the author (1992), Theorem 2:

$$X <_{LH,=} Y \quad \Rightarrow \quad \delta H[X] \leq \delta H[Y]\,. \tag{2.5}$$

Note that the $(LH,=)$-ordering is also a generalization of the k-ordering. This is immediate in view of the equivalence

$$E[LH(X)] = E[LH(Y)] \quad \Leftrightarrow \quad H[Y] = \frac{E[Y]}{E[X]} \cdot H[X]\,, \tag{2.6}$$

always fulfilled in case $H[X] = E[X]$.

3 LH-ordering and stochastic dominance

As shown in Heilmann (1986), it is possible to interpret k-ordering in terms of a stochastic dominance relation between transformed risks. The proposed construction extends to the $(LH,=)$-ordering. Using the relation $F_{LH(X)}(x) = F_X(xH[X])$ between distribution functions, one gets the relation between stop-loss transforms:

$$\pi_{LH(X)}(d) = \frac{1}{H[X]} \cdot \pi_X(dH[X]), \quad d \in \mathbb{R}\,. \tag{3.1}$$

For positive risks $X \geq 0$ consider now a transformed risk $T(X)$ defined by its distribution as follows:

$$F_{T(X)}(x) := 1 - \frac{\pi_X(xH[X])}{E[X]} = 1 - \frac{\pi_{LH(X)}(x)}{E[LH(X)]}\,, \quad x \geq 0\,. \tag{3.2}$$

With $<_{st}$ the usual *stochastic dominance* relation, one can formulate the following characterization of $(LH,=)$-ordering.

Proposition 3.1. Let X, Y be positive risks, $LH(X)$ the loss ratio transform to the risk price $H[X]$ and $T(X)$ the transformed risk defined by (3.2). Then one has the equivalence

$$\begin{aligned} X <_{LH,=} Y \quad &\Leftrightarrow \quad LH(X) <_{sl,=} LH(Y) \\ &\Leftrightarrow \quad T(X) <_{st} T(Y)\,. \end{aligned} \tag{3.3}$$

Note that by (2.6) the mean value principle $H[X] = E[X]$ yields as special case the characterization of the k-ordering obtained by Heilmann (1985/86).

An alternative similar characterization of LH-ordering is obtained as follows. For positive risks $X \geq 0$, let $F^*(x)$ be the distribution function of the transformed risk $X^* = X \cdot E[LH(X)]$. Then one can define a transformed risk $S(X)$ with density function

$$\begin{aligned} f_{S(X)}(x) &= \frac{1}{E[X^*]} \cdot (1 - F^*(x)) \\ &= \frac{H[X]}{E[X]^2} \cdot \left(1 - F_X\left(x \cdot \frac{H[X]}{E[X]}\right)\right), \end{aligned} \tag{3.4}$$

and distribution function

$$F_{S(X)}(x) = 1 - \frac{1}{E[X]} \cdot \pi_{LH(X)}\left(\frac{x}{E[X]}\right). \tag{3.5}$$

As a consequence one obtains the following characterization of LH-ordering.

Proposition 3.2. Let X, Y be positive risks and $S(X)$ the transformed risk defined by (3.5). Under equal mean of risks $E[X] = E[Y]$, the following equivalences hold:

$$X <_{LH} Y \quad \Leftrightarrow \quad LH(X) <_{sl} LH(Y) \quad \Leftrightarrow \quad S(X) <_{st} S(Y). \tag{3.6}$$

The existence of pairs of risks X, Y satisfying the equivalent conditions (3.6) has the following intriguing consequence. The loss ratio stop-loss order implies that $E[LH(X)] \leq E[LH(Y)]$, which under equal mean of risks $E[X] = E[Y]$, implies the pricing ordering relation $H[Y] \leq H[X]$. On the other side (3.6) implies by (2.3) the opposite return ordering relation $BH[X] \leq BH[Y]$. Therefore it might theoretically be possible that risks with lower risk prices leave to risk underwriters higher stable retention ratios. However the existence of such pairs must be settled, and there are examples which suggest their non-existence. Phenomena like this and similar ones are studied in Section 4.

Setting $\mu = E[X] = E[Y]$ and using the formula (3.1), the second decision rule in (3.6) is seen to be equivalent to the inequality

$$\frac{H[Y]}{H[X]} \leq \frac{\pi_Y(\frac{x}{\mu} \cdot H[Y])}{\pi_X(\frac{x}{\mu} \cdot H[X])}, \quad \text{uniformly for all} \quad x \in [0, \infty). \tag{3.7}$$

In the special case of a *mean preserving* pricing principle with the property $E[X] = E[Y]$ implies $H[X] = H[Y]$ (e.g. an expected value principle), the $(LH, =)$-ordering is compatible with the ordinary stop-loss ordering with equal means.

Corollary 3.1. Let X, Y be positive risks and suppose that $E[X] = E[Y]$, $H[X] = H[Y]$. Then the following equivalent conditions hold:

$$\begin{aligned} X <_{LH,=} Y \quad \Leftrightarrow \quad X <_{sl,=} Y \quad &\Leftrightarrow \quad LH(X) <_{sl,=} LH(Y) \\ &\Leftrightarrow \quad T(X) <_{st} T(Y) \,. \end{aligned} \tag{3.8}$$

4 Ordering of risks and LH-ordering

In this Section the effect of ordering of risks between X and Y on loss ratio ordering between $LH(X)$ and $LH(Y)$ and their induced total order relations is exemplified. As classical applications we present non-life insurance situations (rating large claims in reinsurance) and financial situations (valuation of option prices in Black-Scholes model).
Similar results can be obtained for life insurance using the methods discussed in Kling (1993), Chapter 2. As Remark 4.2 suggests, it is even possible to imagine non-trivial applications in "pure" mathematics following the mottoes "Purity in Applications" and "Applied Mathematics is Bad Mathematics" discussed by Poston and Halmos in Steen (1981).

Example 4.1: stop-loss ordered Pareto risks equal in return and stable return order

Consider risks with a distribution from a Pareto (λ, α) family given by

$$F(x) = 1 - \left(\frac{\lambda}{\lambda + x}\right)^{\alpha}, \quad \alpha > 2, \ \lambda > 0, \ x \geq 0 \,. \tag{4.1}$$

The mean, square of coefficient of variation and net stop-loss prices are given by

$$\mu = \frac{\lambda}{\alpha - 1}, \quad k^2 = 2\left(\frac{\alpha - 1}{\alpha - 2}\right) - 1, \quad \pi(d) = \mu \cdot \left(\frac{\lambda}{\lambda + d}\right)^{\alpha - 1} . \tag{4.2}$$

The convention is made that quantities for different risks X, Y are distinguished throughout using indices. For the special Karlsruhe pricing principle $K[X] = (1 + k_X^2)\mu_X$, consider the $(LH, =)$-ordering. For two Pareto risks X, Y the condition $E[LK(X)] = E[LK(Y)]$ is fulfilled exactly when $k_X = k_Y$, that is $\alpha_X = \alpha_Y$. By (3.1) $(LH, =)$-ordering depends upon the sign of the following univariate function on $[0, \infty)$:

$$f_H(x) = \frac{1}{H[X] \cdot H[Y]} \cdot \{H[X] \cdot \pi_Y(xH[Y]) - H[Y] \cdot \pi_X(xH[X])\}. \tag{4.3}$$

In the particular case $H = K$, $\alpha_X = \alpha_Y = \alpha$, $k_X = k_Y = k$, one has

$$f_K\left(\frac{x}{1+k^2}\right) = \frac{1}{1+k^2}\left\{\frac{\pi_Y(x\mu_Y)}{\mu_Y} - \frac{\pi_X(x\mu_X)}{\mu_X}\right\}. \tag{4.4}$$

Setting $f(x) = (1 + k^2) \cdot f_K(\frac{x}{1+k^2})$ one sees that in case $\alpha_X = \alpha_Y = \alpha$:

$$f'(x) = F_Y(x\mu_Y) - F_X(x\mu_X) = 0. \tag{4.5}$$

Since $f(0) = 0$ one must have $f(x) = 0$ on $[0, \infty)$, which implies that X and Y are equal in the $(LK, =)$-ordering. In particular they have identical K-stable retention ratios and K-stable return indices. Using (2.2) one has

$$\delta K[X] = \delta K[Y] = \frac{1}{1+k^2} = \frac{1}{2}\left(\frac{\alpha-2}{\alpha-1}\right), \tag{4.6}$$

and (2.1) shows that $BK := BK[X] = BK[Y]$ is the following non-linear fixed-point (when it exists):

$$BK = \frac{1}{2}\left(\frac{\alpha-2}{\alpha-1}\right) \cdot \left(\frac{\alpha-2}{\alpha-2BK}\right)^{\alpha-1}. \tag{4.7}$$

Whatever the parameter value λ is, by fixed α, all Pareto risks are equal in return and stable return order. However by Van Heerwaarden (1991), p. 87, these Pareto risks are always stop-loss ordered according to the rule

$$X <_{sl} Y \quad \Leftrightarrow \quad \lambda_X \le \lambda_Y. \tag{4.8}$$

This shows the existence of infinitely many arbitrarily stop-loss ordered Pareto risks, which are all equal in total return and stable return order. This result has a significant interpretation for the reinsurance of large claims based on the Pareto model. It is easy to check that α is a deductible invariant parameter, which means that the Pareto family is closed under changes of the deductible (e.g. Daykin et al. (1994), Exercise 3.3.23). Since the parameter λ reflects the change in deductible, the above result says that varying the deductible in a Pareto risk with fixed α changes its Karlsruhe premium, but leaves invariant its K-stable retention ratio and K-stable return index.

Example 4.2: non-existence of loss ratio ordered risks with equal mean of risks

Consider the special pricing principle

$$H[X] = \mu_X + \pi_X(\mu_X), \tag{4.9}$$

which has been considered in a finance context by the author (1991). In forthcoming papers mentioned in the references, the author provides further applications, generalizations and justifications for the use of this theoretical pricing principle.

Proposition 4.1. Let X, Y be two risks with equal mean μ, and let H be the pricing principle defined by (4.9). Then the ordering relation $X <_{LH} Y$ holds if and only if the risks are stop-loss ordered with equal means, that is $X <_{sl,=} Y$, and the risk prices are equal, that is $H[Y] = H[X]$.

Proof. From the comment following Proposition 3.2, loss ratio ordered risks $X <_{LH} Y$ with equal mean of risks can only exist provided $H[Y] \leq H[X]$, that is $\pi_Y(\mu) \leq \pi_X(\mu)$. Denote by $b_X = BH[X]$ the H-stable retention ratio to the pricing principle (4.9). Applying (2.1) one sees that

$$b_X = \frac{\pi_X(\mu)}{\mu + \pi_X(\mu)}. \tag{4.10}$$

Since $\pi_Y(\mu) \leq \pi_X(\mu)$ one must have $b_Y \leq b_X$. By (2.3) if the relation $X <_{LH} Y$ holds, then necessarily $\pi_Y(\mu) = \pi_X(\mu)$, hence $H[Y] = H[X]$. Moreover $X <_{sl,=} Y$ by (1.1). The converse follows immediately from the definition (1.1) of LH-ordering.

Example 4.3: stable retention ratios and prices for lognormal risks with equal mean, which are preserved under stop-loss order.

Consider risks X with a distribution from a lognormal $\ln N(\mu, \sigma)$ family given by

$$F(x) = N\left(\frac{\ln\{x\} - \mu}{\sigma}\right), \tag{4.11}$$

where $N(x)$ is the standard normal distribution. In the framework of the Black-Scholes option pricing model, with initial asset prices standardized to one unit, the one-year accumulated risk-free rate and option prices are given by

$$r = E[X] = \exp\left(\mu + \frac{1}{2}\sigma^2\right), \tag{4.12}$$

$$\begin{aligned} C(d) = E[(X-d)_+] &= rN\left(\frac{\ln\{\frac{r}{d}\}}{\sigma} + \frac{1}{2}\sigma\right) - dN\left(\frac{\ln\{\frac{r}{d}\}}{\sigma} - \frac{1}{2}\sigma\right) \\ &= rN\left(\frac{\mu - \ln\{d\}}{\sigma} + \sigma\right) - dN\left(\frac{\mu - \ln\{d\}}{\sigma}\right). \end{aligned} \tag{4.13}$$

Given two different risks X, Y from this family, equal mean of risks implies a constant risk-free rate and the relation

$$\mu_Y = \mu_X + \frac{1}{2}(\sigma_X^2 - \sigma_Y^2). \tag{4.14}$$

In this finance context the pricing principle (4.9) takes the form

$$H[X] = r + C_X(r) = 2rN\left(\frac{1}{2}\sigma_X\right). \tag{4.15}$$

Let us first clarify when the stop-loss order relation $X <_{sl,=} Y$ holds.

Lemma 4.1. Lognormal distributions with equal mean increase in stop-loss order with increasing volatility parameter.

Proof. Following Van Heerwaarden (1991), Section 6.4, to compare lognormal distributions with equal mean r, substitute $\mu = \ln\{r\} - \frac{1}{2}\sigma^2$ in the lognormal density to obtain

$$f(x;\sigma) = \frac{1}{\sqrt{2\pi}\sigma x} \cdot \exp\left\{-\frac{1}{2}\left(\frac{\ln\{\frac{x}{r}\} + \frac{1}{2}\sigma^2}{\sigma}\right)^2\right\}, \quad x > 0. \tag{4.16}$$

Let us determine the sign changes of $\frac{\partial}{\partial\sigma} f(x;\sigma)$, or equivalently $\frac{\partial}{\partial\sigma} \ln f(x;\sigma)$ on $(0,\infty)$. One has

$$\frac{\partial}{\partial\sigma} \ln f(x;\sigma) = \frac{1}{\sigma}\left\{\frac{\ln\{\frac{x}{r}\}^2}{\sigma^2} - \frac{1}{4}\sigma^2 - 1\right\}. \tag{4.17}$$

Observe that

$$\ln\{x\}^2 = \begin{cases} \ln\left\{\frac{1}{x}\right\}^2, & 0 < x < 1, \text{ is monotone decreasing} \\ \ln\{x\}^2, & x \geq 1, \text{ is monotone increasing.} \end{cases} \tag{4.18}$$

It follows that the expression (4.17) has at most two sign changes on $(0,\infty)$, being a continuous function of x that equals the sum of a constant, a non-decreasing function and a non-increasing function. The result follows applying Theorem 6.3.1 in Van Heerwaarden (1991).

Denote by $b_X = BH[X]$ the H-stable retention ratio to the pricing principle (4.15). From (4.10) one knows that

$$b_X = \frac{C_X(r)}{r + C_X(r)}. \tag{4.19}$$

Using Lemma 4.1 let us state a relevant characterization of the stable retention ratio, which in particular holds in a classical option pricing context.

Proposition 4.2. For lognormal risks with equal mean and pricing principle (4.15), the following conditions are equivalent:

$$X <_{sl,=} Y \quad \Leftrightarrow \quad BH[X] \leq BH[Y] \quad \Leftrightarrow \quad H[X] \leq H[Y] \tag{4.20}$$

Example 4.4: Pareto risks with equal mean, which are less dangerous in $(sl,=)$-order and pricing order, but with higher stable return indices and stable retention ratios.

Let us show that risks and their rating do not always behave like in Proposition 4.2 of Example 4.3. Given two Pareto risks X, Y with equal mean with a distribution (4.1), one knows from Van Heerwaarden (1991), p. 87, that

$$X <_{sl,=} Y \quad \Leftrightarrow \quad \lambda_Y = \left(\frac{\alpha_Y - 1}{\alpha_X - 1}\right)\lambda_X \quad \text{and } \alpha_X \geq \alpha_Y > 2\,. \tag{4.21}$$

Since $1 + k_x^2 = 2(\frac{\alpha_X - 1}{\alpha_X - 2})$ one sees that $k_X \leq k_Y$ whenever $X <_{sl,=} Y$. For Karlsruhe prices $K[X] = (1 + k_X^2) \cdot \mu_X$ with stable return indices $\delta K[X] = (1 + k_X^2)^{-1}$, it follows that (see also author (1992, Example 3)):

$$(X <_{sl,=} Y \quad \Leftrightarrow \quad K[X] \leq K[Y]) \quad \Rightarrow \quad \delta K[X] \geq \delta K[Y]\,. \tag{4.22}$$

Furthermore from (4.7) one knows that the stable retention ratio $b_X := BK[X]$ is zero of the function

$$f_X(d) = d - \frac{1}{2}\left(\frac{\alpha_X - 2}{\alpha_X - 1}\right) \cdot \left(\frac{\alpha_X - 2}{\alpha_X - 2d}\right)^{\alpha_X - 1}. \tag{4.23}$$

From (2.1) one sees that in general $0 < b_X < 1$ (assuming the natural mean exceeding property $H[X] > E[X]$). In case $\alpha_Y \leq \alpha_X$ one has

$$\frac{\alpha_Y - 2}{\alpha_Y - 2d} \leq \frac{\alpha_X - 2}{\alpha_X - 2d} \qquad \text{for all } 0 < d < 1\,. \tag{4.24}$$

It follows that $f_X(d) \leq f_Y(d)$ for all $0 < d < 1$. But then the zeros of f_X, f_Y relate as $b_X \geq b_Y$, which shows the implication

$$X <_{sl,=} Y \quad \Rightarrow \quad BK[X] \geq BK[Y]\,. \tag{4.25}$$

Remark 4.1. The practical meaning of the qualitatively opposite behaviour of the stable retention ratio in Example 4.3 and 4.4 can be appreciated in their proper context. Proposition 4.2, of use in option pricing, reflects the fact that a rate of return on financial assets (by constant risk-free rate) has to increase by increasing volatility. On the other side the relation (4.23), of use in reinsurance of large claims, means that (by fixed mean) large claims from a Pareto distribution with Karlsruhe prices, which are less dangerous in stop-loss order, will provide a greater stable retention ratio on insurance liabilities.

Remark 4.2. Suppose given the situation of Example 4.3. A direct proof based on (3.7) of the fact that X, Y cannot be LH-ordered when $H[Y] \neq H[X]$, as in Example 4.2, would require complicated and non-trivial analytical calculations. Instead of this our indirect proof of Proposition 4.1 can be used to get an "ordering of risks" proof of the needed non-trivial probabilistic inequality.

Corollary 4.1. Let X, Y be two lognormal distributed random variables with equal means such that $\mu_Y - \mu_X = \frac{1}{2}(\sigma_X^2 - \sigma_Y^2) \geq 0$. Further let x_0, γ be defined by the condition

$$\gamma := \frac{\ln\{x_0 N(\frac{1}{2}\sigma_Y)\} - \mu_Y}{\sigma_Y} = \frac{\ln\{x_0 N(\frac{1}{2}\sigma_X)\} - \mu_X}{\sigma_X}. \tag{4.26}$$

Then one has the inequality

$$\frac{N(\frac{1}{2}\sigma_Y)}{N(\frac{1}{2}\sigma_X)} \geq \frac{N(\gamma + \sigma_Y)}{N(\gamma + \sigma_X)}, \tag{4.27}$$

where equality holds exactly when $\sigma_Y = \sigma_X$.

Proof. Based on (4.15) and using (3.7) one has $X <_{LH} Y$ if and only if

$$\frac{N(\frac{1}{2}\sigma_Y)}{N(\frac{1}{2}\sigma_X)} \geq \frac{C_Y(2xN(\frac{1}{2}\sigma_Y))}{C_X(2xN(\frac{1}{2}\sigma_X))}, \quad \text{uniformly for all } x \geq 0. \tag{4.28}$$

This inequality is fulfilled for $x = 0$, and thus also for sufficiently small $x > 0$ by continuity. However since X, Y cannot be LH-ordered, one knows that (4.28) cannot hold uniformly for all $x \geq 0$. To simplify notations set

$$\alpha = N\left(\frac{1}{2}\sigma_X\right), \quad \beta = N\left(\frac{1}{2}\sigma_Y\right), \tag{4.29}$$

and consider the function

$$f(x) = \beta C_X(\alpha x) - \alpha C_X(\beta x). \tag{4.30}$$

Then the non-validity of (4.28) reflects the fact that the stationary point $f'(x_0) = 0$ yields a positive local minimum of $f(x)$. Since $C_X'(x) = N(\frac{\ln\{x\} - \mu_X}{\sigma_X})$ -1 one sees that $f'(x_0) = 0$ exactly when the technical condition (4.26) holds. Using the second formula in (4.13) it follows that

$$f(x_0) = r\{\beta N(\gamma + \sigma_X) - \alpha N(\gamma + \sigma_Y)\} \geq 0, \tag{4.31}$$

which is the desired inequality.

5 Ordering characterization of a generalized Dutch premium principle

In the present Section a generalized form of the Dutch premium principle is characterized by the property of no unjustified loading and order preserving properties. The result allows for a mathematical classification of some perfectly hedged bonus strategies such that the associated insurance premiums are consistent with the required properties. In particular we fill the gap (1995c), Proposition 3.1, and complete in a qualitative way the results obtained previously (1994b).
The recent developments in Van Heerwaarden/Kaas (1992) and the author (1994b) have led to the following meaningful two-parametric premium principle:

$$\begin{aligned} H[X] &= \left(1 - \frac{1}{2}\theta|1-\beta|\right)E[X] + \frac{1}{2}\theta E[\,|X-\beta E[X]|\,] \\ &= \begin{cases} E[X] + \theta E[(\beta E[X] - X)_+], & \theta \geq 0,\ 0 \leq \beta \leq 1\,, \\ E[X] + \theta E[(X - \beta E[X])_+], & \theta \geq 0,\ \beta \geq 1\,, \end{cases} \end{aligned} \tag{5.1}$$

where X is a random variable such that (5.1) is well-defined. In the mentioned papers it has been shown that this premium principle gives no unjustified loading, that is $H[c] = c$ for any constant $c \geq 0$, and preserves stochastic dominance and stop-loss order provided $\theta \leq 1$. In fact the stronger *necessary* and sufficient condition characterizes this attractive premium principle.

Theorem 5.1. The following statements hold:
(a) The premium principle

$$H[X] = E[X] + \theta E[(\beta E[X] - X)_+], \quad \theta \geq 0,\ \beta \geq 0\,, \tag{5.2}$$

gives no unjustified loading, preserves stochastic dominance and stop-loss order if and only if $0 \leq \theta \leq 1$, $0 \leq \beta \leq 1$.
(b) The premium principle

$$H[X] = E[X] + \theta E[(X - \beta E[X])_+], \quad \theta \geq 0,\ \beta \geq 0\,, \tag{5.3}$$

gives no unjustified loading, preserves stochastic dominance and stop-loss order if and only if $0 \leq \theta \leq 1$, $\beta \geq 1$.

Proof. It is immediately seen that the property of no unjustified loading is fulfilled if and only if $\beta \leq 1$ (case (a)) respectively $\beta \geq 1$ (case (b)). It has been shown by Van Heerwaarden/Kaas (1992) and the author (1994c) that the condition $\theta \leq 1$ is sufficient to guarantee the stated order preserving properties. To prove that $\theta \leq 1$ is a necessary condition, it suffices to construct stochastically ordered random variables $X \leq_{st} Y$ such that $H[Y] < H[X]$ in case $\theta > 1$, that is stochastic dominance is not preserved. In the following let $\mu_X = E[X]$ be the mean and $F_X(x) = \Pr(X \leq x)$ the distribution of X. Recall that $X \leq_{st} Y$ if $F_X(x) \geq F_Y(x)$ for all x.

Case (a): $\theta > 1,\ 0 < \beta \leq 1$

Consider a 3-atomic distribution of the form

$$F_X(x) = \begin{cases} 0, & x < \beta a\,, \\ p, & \beta a \leq x < \beta\mu_Y\,, \\ q, & \beta\mu_Y \leq x < \beta b\,, \\ 1, & x \geq \beta b\,, \end{cases} \tag{5.4}$$

and a 2-atomic distribution of the form

$$F_Y(x) = \begin{cases} 0, & x < \beta\mu_Y\,, \\ q, & \beta\mu_Y \leq x < \beta b\,, \\ 1, & x \geq \beta b\,. \end{cases} \tag{5.5}$$

One looks for parameter values $0 < p < q < 1$, $a \leq \mu_X < \mu_Y \leq b$, which imply $F_X(x) \geq F_Y(x)$ for all x, and such that

$$H[Y] - H[X] = (\mu_Y - \mu_X)(1 - \theta + p\beta\theta) < 0\,. \tag{5.6}$$

The choice

$$\begin{aligned} &0 < p < \min\left\{\frac{\theta - 1}{\beta\theta}, 1\right\}, \quad p < q < 1\,, \\ &a < (1-q)\beta c \leq b = (1 - q\beta)c, \quad c > 0\,, \end{aligned} \tag{5.7}$$

implies $a \leq \mu_X = p\beta a + (1 - p\beta)\mu_Y < \mu_Y = (1-q)\beta c \leq b$, and (5.6) is fulfilled.

Case (b): $\theta > 1,\ \beta \geq 1$

Consider 2-atomic distributions of the form

$$F_X(x) = \begin{cases} 0, & x < \beta a\,, \\ p, & \beta a \leq x < \beta\mu_Y\,, \\ 1, & x \geq \beta\mu_Y\,, \end{cases} \tag{5.8}$$

$$F_Y(x) = \begin{cases} 0, & x < \beta b\,, \\ p, & \beta b \leq x < \beta\mu_Y\,, \\ 1, & x \geq \beta\mu_Y\,, \end{cases} \tag{5.9}$$

where one needs parameter values $0 < p < 1$, $a \leq \mu_X < b \leq \mu_Y$, which imply $F_X(x) \geq F_Y(x)$ for all x, such that

$$H[Y] - H[X] = (\mu_Y - \mu_X)(1 - (1-p)\beta\theta) < 0\,. \tag{5.10}$$

One shows that the choice

$$\begin{aligned} &0 \leq 1 - \frac{1}{\beta} < p < 1 - \frac{1}{\beta\theta}, \quad b = (1-(1-p)\beta)c, \quad c > 0\,, \\ &a < \min\left\{c, \frac{(1-p)\beta}{(1-p\beta)} \cdot p\beta c\right\}, \end{aligned} \tag{5.11}$$

satisfies $a \leq \mu_X < b \leq \mu_Y$ and (5.10). This completes the proof of our ordering characterization of the generalized Dutch premium principle.

The above Theorem can be applied to design bonus strategies with *consistent* insurance premiums, and permits to complete some of our previous results in this area. Let X be an insurance risk and assume the insurance premium $P = H[X]$ gives no unjustified loading and preserves stochastic dominance and stop-loss order. The following class of *perfectly hedged bonus strategies* has been introduced by the author (1994b). For the purpose of this paper a bonus payment $D = D[X] \geq 0$ is said to be perfectly hedged if there exists a constant $d > 0$ and a reinsurance form $Z = Z[X]$, which may be a risk-exchange, such that with probability one:

$$d + Z = X + D\,. \tag{5.12}$$

Then the insurance premium to the liability $X + D$ is necessarily given by

$$P = H[X] = d + H^R[Z]\,, \tag{5.13}$$

where $H^R[\cdot]$ is the premium principle of the reinsurer. For simplicity let us assume that $H^R[Z] = (1+\theta)E[Z]$, $\theta \geq 0$, is the expected value principle. To illustrate consider first a bonus perfectly hedged by a pure stop-loss cover. In this situation one has $Z = (X - d)_+$, $D = (d - X)_+$. Setting $d = \alpha E[X]$, $\alpha > 0$, one finds that $H[X] = \alpha E[X] + (1+\theta)E[(X-d)_+]$ gives no unjustified loading if one of the following two cases occur: case I if $\theta = 0$, $\alpha < 1$, case II if $\theta \geq 0$, $\alpha = 1$. In case I the identity (5.12) yields $H[X] = E[X] + E[(\alpha E[X] - X)_+]$ which is known to preserve the required ordering properties. In case II the premium functional $H[X] = E[X] + (1+\theta)E[(X - E[X])_+]$ preserves the ordering properties only if $\theta = 0$. This fact, not shown in (1994b), is a direct consequence of the above Theorem. It follows that the class of consistent bonus formulas perfectly hedged by a pure stop-loss cover is characterized by $\theta = 0$ and $D = (\alpha E[X] - X)_+$, $0 < \alpha \leq 1$. Since $\theta = 0$ leads to a "technical ruin" of reinsurance companies, a bonus formula of this type cannot be recommended in case decisions are based on the properties of no unjustified loading and orderings preservation.
However in case $\theta > 0$ consistent bonus formulas $D = g(\alpha E[X] - X)_+$ can be obtained using the perfect hedge

$$Z = g(X - \alpha E[X])_+ + (1-g)(X - E[X]) - \begin{cases} g(1-\alpha)E[X], & 0 < \alpha \leq 1\,, \\ g(\frac{\alpha-1}{\theta})E[X], & \alpha \geq 1\,, \end{cases}$$
$$g = \frac{\theta}{1+\theta}, \quad d = (1 + (\alpha - 1)_+)E[X]\,. \tag{5.14}$$

A straightforward calculation shows that

$$H[X] = \begin{cases} E[X] + \theta E[(\alpha E[X] - X)_+], & 0 \leq \alpha \leq 1\,, \\ E[X] + \theta E[(X - \alpha E[X])_+], & \alpha \geq 1\,. \end{cases} \tag{5.15}$$

Our Theorem implies that consistent bonus formulas of this type can only occur for $0 < \theta \leq 1$. For example, choosing $g = \theta/(1+\theta) > 1/2$, $\theta > 1$, leads to inconsistent insurance premiums. Other remarks concerning this bonus strategy are given by the author (1994b).

6 Ordering properties of splitting premium principles

The author (1994a) has developed some interest in the following premium calculation principle. Given is a random variable representing a risk, which can be split up into two components $X_1 = f(X)$, $X_2 = g(X)$, transformed random variables of X, such that $X = X_1 + X_2$. In case a variance principle $H[\cdot] = E[\cdot] + \theta \operatorname{Var}[\cdot]$ is applied, and the components are insured separately, the splitting risk premium needed to cover the risk X equals

$$H[X] = E[X] + \theta R_X(f, g), \tag{6.1}$$

where

$$R_X(f, g) = \operatorname{Var}[f(X)] + \operatorname{Var}[g(X)] \tag{6.2}$$

is called the total variance splitting risk of the insurance risk X. It is well-known that many of the common premium calculation principles, for example the variance and the standard deviation principle, do not preserve stochastic dominance, and they preserve stop-loss order only by equal means (e.g. Van Heerwaarden (1991), Section 8.3). It follows that these premium principles are inconsistent with the preferences of risk averse decision makers with an arbitrary non-decreasing utility function (e.g. Goovaerts et al. (1990)). The same disadvantage is shared by the above splitting premium principle.

Proposition 6.1. Let X be a risk and let $f(x)$, $g(x)$ be two differentiable real functions such that $f(x) + g(x) = x$. Then the splitting premium principle $H[X] = E[X] + \theta R_X(f, g)$ does not preserve stochastic dominance.

Proof. One has to show that there exists two stochastically ordered risks $X \le_{st} Y$ such that $H[Y] < H[X]$. Let $Y \equiv c > 0$ be a constant degenerate risk and X a diatomic risk with support $\{0, c\}$ such that $0 < p = \Pr(X = c) < 1$, $q = 1 - p = \Pr(X = 0)$. Then one has $X \le_{st} Y$, $E[Y] = c$, $E[X] = cp$, and $\operatorname{Var}[f(X)] = pq\{f(c) - f(0)\}^2$, $\operatorname{Var}[g(X)] = pq\{g(c) - g(0)\}^2$. Since $f(x) + g(x) = x$ one has $g(c) - g(0) = c - \{f(c) - f(0)\}$. Furthermore by the mean value principle of real analysis, there exists $c_0 \in (0, c)$ such that $f(c) - f(0) = f'(c_0)c$. After straightforward calculation one gets $H[Y] - H[X] = cq\{1 - cp\theta\Delta_0\}$, with $\Delta_0 = f'(c_0)^2 + \{1 - f'(c_0)\}^2 > 0$. Therefore any choice $c > (p\theta\Delta_0)^{-1}$ does the job.

Despite the mentioned disadvantage, a splitting premium principle might nevertheless be useful in some specific insurance economy of risk-averse agents. As an illustration let us construct a *second degree stop-loss order preserving* splitting premium principle for an economic environment of (positive) risks with *fixed* mean μ, variance σ^2 and net stop-loss premium $\pi(d)$ to a given deductible d. This particular problem has been suggested to us by Briys (1990), p. 37, which states that no application of this kind seems to exist in the literature on microeconomic theory of the demand of insurance. Indeed it is known that a second degree stop-loss order preserving functional is consistent with the common preferences of a group of decision makers with a *decreasing risk aversion* (criterion of Whitmore (1970)). For a proof consult for example the new monograph by Kaas et al. (1994), Theorem 2.1 in chapter V, or Van Heerwaarden (1991), chapter 5. In practice this criterion is useful because larger (re)insurers tend to charge smaller (re)insurance premiums, in accordance with a decreasing risk aversion with wealth (e.g. Van Heerwaarden (1991), p. 112).

Example 6.1: A linear combination of proportional and stop-loss reinsurance defines a two-component splitting of risks with

$$f(x) = r \cdot (x - (x-d)_+)\,, \tag{6.3}$$
$$g(x) = (1-r) \cdot x + r \cdot (x-d)_+\,, \tag{6.4}$$

where r is the proportional retention and d the stop-loss deductible. Setting $\pi(d) = E[(X-d)_+]$, $\bar{\pi}(d) = E[(d-X)_+] = d-\mu+\pi(d)$, $\pi_2(d) = E[(X-d)_+^2]$, one gets after calculation

$$\begin{aligned} R_X(f,g) = \{r^2 + (1-r)^2\}\sigma^2 + 2r(1-2r)\pi(d)\bar{\pi}(d) \\ + 2r(1-r)\{\pi_2(d) - \pi(d)^2\}\,. \end{aligned} \tag{6.5}$$

In view of this formula it is obvious that the splitting premium principle $H[X] = E[X] + \theta R_X(f,g)$ preserves the stop-loss order of degree two, that is $X \le_{(2)} Y$ implies $H[X] \le H[Y]$, providing rating of risks is restricted to the space of random variables

$$\begin{aligned} &D(\mu,\sigma,\pi(d)) \\ &\quad := \{X : E[X] = \mu, \mathrm{Var}[X] = \sigma^2, E[(X-d)_+] = \pi(d)\}\,, \end{aligned} \tag{6.6}$$

which represents a set of risks with fixed mean, variance and net stop-loss premium. In this respect it is interesting to mention that there exists an extremal diatomic distribution which maximizes the second order stop-loss moment $\pi_2(d)$ over the space of risks $D(\mu, \sigma, \pi(d))$, a fact which finds herewith a further application. This result has been discovered by Schmitter (1993) (see author (1994c) and Schmitter (1995)).

The actuarial importance of the discussed issue is best illustrated using our previous simple example.

Example 6.1 (continued)**:** rating long tailed risks
Let X be a compound geometric exponential claim with tail distribution $\overline{F}_X(x) = \alpha \exp\{-\frac{x}{\beta}\}$, $x > 0$, $0 < \alpha < 1$, $\beta > 0$, and let Y be a Pareto claim with $\overline{F}_Y(x) = (\frac{a+x}{a})^{-\gamma}$, $x \geq 0$, $a > 0$, $\gamma > 2$. The following risk characteristics are calculated:

$$\begin{aligned} &\mu_X = \alpha\beta, \quad \sigma_X^2 = \alpha(2-\alpha)\beta^2, \\ &\pi_X(d) = \alpha\beta \exp\left\{-\frac{d}{\beta}\right\}, \quad \pi_{2,X}(d) = 2\alpha\beta^2 \exp\left\{-\frac{d}{\beta}\right\}. \end{aligned} \tag{6.7}$$

$$\begin{aligned} &\mu_Y = \frac{a}{\gamma-1}, \quad \sigma_Y^2 = \frac{\gamma a^2}{(\gamma-1)^2(\gamma-2)}, \\ &\pi_Y(d) = \mu_Y \left(\frac{a}{a+d}\right)^{\gamma-1}, \quad \pi_{2,Y}(d) = \frac{2a^2}{(\gamma-1)(\gamma-2)} \left(\frac{a}{a+d}\right)^{\gamma-2}. \end{aligned} \tag{6.8}$$

Now suppose both risks belong to the class of risks $D(\mu, \sigma, \pi(d))$, which is the case provided the parameters satisfy the following relations:

$$\alpha = \frac{\gamma-2}{\gamma-1}, \quad \beta = \frac{a}{\gamma-2}, \quad \frac{d}{\beta} = (\gamma-1)\ln\left\{\frac{a+d}{a}\right\}. \tag{6.9}$$

If risk premiums are set according to a mean-variance principle or a generalized Dutch premium principle (5.1), which take into account only μ, σ or $\mu, \pi(d)$, then both risks will have the same risk premium. However it is felt that most (re)insurers would assign to Y a higher premium, because a Pareto distribution generates a longer right tail. Relying only on the risk characteristics $\mu, \sigma, \pi(d)$ implies that an actuary will tend to overprice X and underprice Y. To discriminate between both prices a more appropriate premium principle must be chosen. As an example our simple splitting premium principle will do. Indeed in case $X, Y \in D(\mu, \sigma, \pi(d))$, one has

$$\pi_{2,X}(d) = \left(\frac{2a}{\gamma - 2}\right)\pi(d) < \left(\frac{2(a+d)}{\gamma - 2}\right)\pi(d) = \pi_{2,Y}(d), \quad d > 0\,, \tag{6.10}$$

which implies that $H[X] < H[Y]$.

References

Bowers, N.L. (1969). An upper bound for the net stop-loss premium. Transactions of the Society of Actuaries XIX, 211–18.

Briys, E. (1990). Demande d'assurance et microéconomie de l'incertain. Collection "Finance", Presses Universitaires de France.

Bühlmann, H., Gagliardi, B., Gerber, H.U., Straub, E. (1977). Some inequalities for stop-loss premiums. ASTIN Bulletin 9, 75–83.

Daykin, C.D., Pentikäinen, T., Pesonen, M. (1994). Practical Risk Theory for Actuaries. Monographs on Statistics and Applied Probability 53. Chapman and Hall.

Fishburn, P.C. (1984). Foundations of Risk Measurement. I: Risk as Probable Loss. Management Science 30, 396–406.

Garrido, J. (1993). Discussion of the paper by Ramsay (1993).

Goovaerts, M., DeVylder, F., Haezendonck, J. (1984). Insurance Premiums. North-Holland, Amsterdam.

Goovaerts, M.J., Kaas, R., Heerwaarden, Van A.E., Bauwelinckx, T. (1990). Effective Actuarial Methods. North-Holland, Amsterdam.

Heerwarden, Van A.E. (1991). Ordering of risks: theory and actuarial applications. Ph.D. Thesis, Tinbergen Institute Research Series no. 20, Amsterdam.

Heerwaarden, Van A.E., Kaas, R. (1992). The Dutch premium principles. Insurance: Mathematics and Economics 11, 129–33.

Heilmann, W.-R. (1985). Transformations of claim distributions. Bulletin of the Swiss Association of Actuaries, 57–69.

Heilmann, W.-R. (1986). Ordering of distributions and risk measurement. Blätter der Deutschen Gesellschaft für Versicherungsmathematik 18, 225–35.

Heilmann, W.-R. (1988). Fundamental of Risk Theory. Verlag Versicherungswirtschaft, Karlsruhe.

Hürlimann, W. (1991). Absicherung des Anlagerisikos, Diskontierung der Passiven und Portfoliotheorie. Bulletin of the Swiss Association of Actuaries, 217–50.

Hürlimann, W. (1992). Ordering of risks through loss ratios. Insurance: Mathematics and Economics 11, 49–54.

Hürlimann, W. (1994a). Splitting risk and premium calculation. Bulletin of the Swiss Association of Actuaries, 167–97.

Hürlimann, W. (1994b). A note on experience rating, reinsurance and premium principles. Insurance: Mathematics and Economics 14, 197–204.

Hürlimann, W. (1994c). From the inequalities of Bowers, Kremer and Schmitter to the total stop-loss risk. XXV-th ASTIN Colloquium, Cannes, September 1994.

Hürlimann, W. (1995a). Links between premium principles and reinsurances. International Congress of Actuaries, Brussels, Vol. 2, 141–167.
Hürlimann, W. (1995b). A distribution-free all-finance binomial valuation model. International Congress of Actuaries, Brussels, Vol. 3, 365–382.
Hürlimann, W. (1995c). On fair premium principles and Pareto optimal risk neutral portfolio valuations. International Congress of Actuaries, Brussels, Vol. 1, 189–208.
Hürlimann, W. (1995d). CAPM, derivative pricing and hedging. 5-th AFIR Colloquium, Brussels, September 1995.
Kaas, R., Heerwaarden, Van A.E., Goovaerts, M.J. (1994). Ordering of Actuarial Risks, CAIRE Education Series 1, Brussels.
Kling, B. (1993). Life insurance, a non-life approach. Ph.D. Thesis, Tinbergen Institute Research Series no. 60, Amsterdam.
Mosler, K.C., Scarsini, M. (1993). Stochastic orders and applications: a classified bibliography. Lecture notes in economics and mathematical systems 401.
Ramsay, C.M. (1993). Loading gross premiums for risk without using utility theory. Transactions of the Society of Actuaries XLV, 305–36. Discussion, 337–49.
Rothschild, M., Stiglitz, J.E. (1970). Increasing risk: I. A definition. Journal of Economic Theory 2, 225–43.
Rothschild, M., Stiglitz, J.E. (1971). Increasing risk: II. Its economic consequences. Journal of Economic Theory 3, 66–84.
Rothschild, M., Stiglitz, J.E. (1972). Addendum to 'Increasing risk: I. A definition'. Journal of Economic Theory 5, 306.
Schmitter, H. (1995). An upper limit of the stop-loss variance. XXVI-th ASTIN Colloquium, Leuven, September 1995.
Shaked, M., Shanthikumar, J.G. (1994). Stochastic Orders and Their Applications. Academic Press, New York.
Steen, L.A. (1981). (Editor) Mathematics Tomorrow. Springer-Verlag.
Whitmore, G.A. (1970). Third degree stochastic dominance. American Economic Review 60, 457–59.

Werner Hürlimann
Allgemeine Mathematik
Winterthur-Leben
Paulstr. 9
CH–8401 Winterthur

Summary

The present paper contributes to the actuarial subjet of "risk measures" as defined in Ramsay (1993) and the subsequent Discussion. Two "relative" risk measures, derived previously by the author from a loss ratio ordering, are reconsidered in the light of some new insight and applications. It is observed that the loss ratio ordering is a generalization of the k-ordering by Heilmann (1985/86). Then loss ratio orderings are interpreted in terms of stochastic dominance relations between transformed risks. Several examples of central interest in non-life (rating large claims in reinsurance) and option pricing in the Black-Scholes model are presented. Concerning "absolute" risk measures and their associated premium loadings, the generalized Dutch premium principle is characterized by the properties of no unjustified loading and stop-loss order preservation. This result allows for a mathematical classification of some perfectly hedged bonus strategies. Finally an example of a second degree stop-loss order preserving splitting premium principle is constructed for the purpose of solving a question suggested by Briys (1990).

Zusammenfassung

Die vorliegende Arbeit liefert einen Beitrag zum Thema "Risikomasse" in der Versicherungswissenschaft, wie es in Ramsay (1993) und der anschliessenden Diskussion definiert wird. Zwei "relative" Risikomasse, die der Autor von einer Schadensatz-Ordnung abgeleitet hat, werden im Licht neuer Einsicht und Anwendungen nochmals betrachtet. Es wird festgestellt, dass die Schadensatz-Ordnung eine Verallgemeinerung der k-Ordnung von Heilmann (1985/86) ist. Anschliessend werden Schadensatz-Ordnungen mit Hilfe von stochastischen Dominanzrelationen zwischen transformierten Risiken interpretiert. Mehrere Beispiele von zentraler Bedeutung in der Nicht-Lebensversicherung (Tarifierung von Grossschäden in der Rückversicherung) und in der Optionspreistheorie von Black-Scholes werden vorgestellt. Im Zusammenhang mit "absoluten" Risikomassen und ihre zugeordnenten Sicherheitszuschläge, wird das verallgemeinerte Dutch Prämienprinzip durch folgende Eigenschaften charakterisiert. Ein Prämienprinzip soll keinen unbegründeten Sicherheitszuschlag enthalten und die Stop-loss Ordnung erhalten. Dieses Resultat erlaubt es, eine mathematische Klassifizierung von einigen perfekt abgesicherten Bonus Strategien zu betrachten. Zum Schluss wird ein Beispiel eines Prämienprinzips konstruiert, das die Stop-Loss Ordnung zweiten Grades erhält. Dies löst eine Fragestellung von Briys (1990).

Résumé

Le présent travail fournit une contribution au sujet actuariel des "mesures du risque" comme définies dans Ramsay (1993) et la Discussion qui suit. Deux mesures du risque "relatives", dérivées précédemment par l'auteur à l'aide d'une relation d'ordre sur taux de sinistre, sont inspectées par rapport à de nouvelles applications. On observe que la relation d'ordre sur taux de sinistre est une généralisation de la relation k-ordre par Heilmann (1985/86). Ensuite on interprète des relations d'ordre sur taux de sinistre à l'aide de relations de dominance stochastique entre risques transformés. Plusieurs exemples d'un intérêt primordial pour

l'assurance non-vie (tarification des grands sinistres en réassurance) et le modèle de Black-Scholes en théorie des options sont présentés. En ce qui concerne les mesures du risque "absolues" et leurs primes majorées correspondantes, on caractérise le principe de calcul des primes Dutch généralisé à l'aide des propriétés suivantes. Un principe de calcul des primes ne doit pas contenir de majoration non justifiée et préserver la relation d'ordre stop-loss. Ce résultat permet de considérer une classification mathématique de quelques stratégies bonus parfaitement couvertes. Finalement on construit un principe de calcul des primes qui préserve la relation d'ordre stop-loss du deuxième degré. Ceci permet de répondre à une question de Briys (1990).