Zeitschrift: Mitteilungen / Schweizerische Aktuarvereinigung = Bulletin / Association Suisse des Actuaires = Bulletin / Swiss Association of Actuaries

Herausgeber: Schweizerische Aktuarvereinigung

Band: - (1996)

Heft: 1

Artikel: Bühlmann's credibility premium in the Bühlmann-Straub model

Autor: Dannenburg, Dennis

DOI: https://doi.org/10.5169/seals-551009

DENNIS DANNENBURG, Amsterdam

Bühlmann's Credibility Premium in the Bühlmann-Straub Model

1 Introduction

One of the main activities of an actuary is to calculate premiums for insured risks. Often it turns out that in order to predict future risks optimally, a premium structure is needed that is rather differentiated. However, a too differentiated structure may not be desirable since premium differentiation is in fact against one of the basic principles of insurance: solidarity among the insured. Also, the variability of the total premium income may be higher than is acceptable for the management of an insurance company.
In this context the Bühlmann-Straub (1970) credibility model will be considered here. This model is an extension of the classical Bühlmann (1967, 1969) model, in the sense that it is possible to take differences into account between the risk exposures underlying the observed risks by the specification of certain weights. These weights may arise in a natural way: if the observed risks are averages of risks with the same characteristics, for instance. Unfortunately, that is not always the case and then one often has to confine oneself with the use of proxy weights, or one simply ignores them. In the latter case the classical Bühlmann model is used even though the claims have been generated from the Bühlmann-Straub model. Each contract in the Bühlmann-Straub model has its own credibility factor, whereas Bühlmann's premiums are based on a credibility factor that is identical for all contracts. Using a constant credibility factor in the Bühlmann-Straub model leads to a more stable premium income and more solidarity among the insured.
The purpose of this paper is to examine some properties of the classical Bühlmann premium if it is used in the Bühlmann-Straub model. In Section 2 the expected values of some frequently used estimators for the parameters in the Bühlmann-Straub model are derived if they are based on general incorrect weights, including the case in which proxy weights are used. If the incorrect weights are all equal to one, the classical Bühlmann model is applied mistakenly. In Section 3 it will be derived that in this case the premiums based on the estimators for the parameters are approximately

equal to the optimal 'classical-Bühlmann-like' premiums in the Bühlmann-Straub model. After that, it is shown how the obtained results for balanced data sets can be extended to data sets that are unbalanced. Section 4 is devoted to the asymptotical justification of the approximation used. An application to automobile insurance data in Section 5 illustrates the results of this paper. Section 6 concludes.

2 The Bühlmann-Straub model and estimators for the parameters

In this section the classical credibility model of Bühlmann and the Bühlmann-Straub model will be described briefly, together with some well-known estimators for the occurring parameters. At the end of this section the expected values of those estimators are derived if they are based on wrong weights.

We consider an insurance portfolio consisting of J contracts. For each contract the realizations are available of risks concerning a certain number of periods of observation. We first assume that the set of observed risks is balanced, i.e. for each contract the same number of T risks has been observed. Of course, such a set of data does usually not occur in practical applications and therefore in Section 3 the obtained results are extended to an unbalanced set of observations.

2.1 The classical Bühlmann model and the Bühlmann-Straub model

The observed risk corresponding to contract j and period of observation t is represented by the random variable X_{jt} ($j \in \{1, \ldots, J\}$, $t \in \{1, \ldots, T\}$). The risks of contract j depend on the not observable risk parameter Θ_j. In the classical Bühlmann model the following assumptions are made:

(i) The sets $\{X_{j1}, \ldots, X_{jT}, \Theta_j\}$ are independent.
(ii) $E[X_{jt}|\Theta_j] = \mu(\Theta_j)$ for all t, for a certain function $\mu(\cdot)$.
(iii) $E\big[\text{Cov}[X_{jt}, X_{ju}|\Theta_j]\big] = \delta_{tu}s^2$. Here $\delta_{tu} = 1$ if $t = u$ and zero otherwise.
(iv) The random variables Θ_j are identically distributed.

The parameter s^2 is a measure for the within-variance in the contracts. Besides this parameter the overall mean $m = E[\mu(\Theta_j)]$, and the parameter for the between-variance $a = \mathrm{Var}[\mu(\Theta_j)]$ appear in Bühlmann's credibility premium for contract j:

$$P_j^B = z\overline{X}_j + (1-z)m\,. \tag{1}$$

Here $\overline{X}_j$ is the average of $X_{j1}, \ldots, X_{jT}$ and the credibility factor is $z = aT/(aT + s^2)$.

In the Bühlmann-Straub model the classical Bühlmann model is extended by scaling the expected value of the conditional variance of X_{jt} by a weight w_{jt}. Thus, assumption (iii) is replaced in the Bühlmann-Straub model with

$$\text{(iii}'\text{)} \qquad E\big[\,\mathrm{Cov}[X_{jt}, X_{ju}|\Theta_j]\big] = \delta_{tu}s^2/w_{jt}\,.$$

The credibility premium for the j-th contract is then equal to

$$P_j^{BS} = z_j X_{jw} + (1-z_j)m\,, \tag{2}$$

with the credibility factor

$$z_j = aw_{j\Sigma}/(aw_{j\Sigma} + s^2)\,, \qquad (w_{j\Sigma} = \Sigma_t w_{jt}) \tag{3}$$

and the weighted average of the observed risks

$$X_{jw} = \sum_{t=1}^{T} \frac{w_{jt}}{w_{j\Sigma}} X_{jt}\,. \tag{4}$$

A possible estimator for m is the weighted overall average

$$M = X_{ww} = \sum_{j=1}^{J} \frac{w_{j\Sigma}}{w_{\Sigma\Sigma}} X_{jw}\,, \qquad (w_{\Sigma\Sigma} = \Sigma_j w_{j\Sigma}) \tag{5}$$

and for s^2 and a the following well-known unbiased estimators will be used (we assume that $T \geq 2$):

$$S^2 = \frac{1}{J(T-1)} \sum_{j=1}^{J}\sum_{t=1}^{T} w_{jt}(X_{jt} - X_{jw})^2 \tag{6}$$

$$A = \frac{w_{\Sigma\Sigma}}{w_{\Sigma\Sigma}^2 - \Sigma_j w_{j\Sigma}^2} \left(\sum_{j=1}^{J} w_{j\Sigma}(X_{jw} - X_{ww})^2 - (J-1)S^2 \right). \tag{7}$$

The statistic X_{ww} is not the optimal linear estimator for m, except in the classical Bühlmann model. The minimum variance linear unbiased estimator depends on s^2 and a (see Goovaerts et al., 1990, p. 153). For a there also exists a pseudo-estimator (see De Vylder, 1981) which is a function of s^2 and a itself. Since parameters for the within- and between-variance are usually not known these estimators are approximated by substitution of the estimates of s^2 and a. It is difficult to derive the expected value of the resulting statistics and therefore such estimators will not be considered in this article.

2.2 *The expected value of the estimators in case of incorrect weights*

Suppose that not the weights w_{jt} are used in the Bühlmann-Straub model, but the incorrect weights $w^\bullet_{jt}$. This implies that the wrong weighted average of the observed risks is used in the credibility premiums, and also that the estimators for m, s^2, and a are different:

$$M^\bullet = X^\bullet_{ww} = \sum_{j=1}^{J} \frac{w^\bullet_{j\Sigma}}{w^\bullet_{\Sigma\Sigma}} X^\bullet_{jw} = \sum_{j=1}^{J}\sum_{t=1}^{T} \frac{w^\bullet_{jt}}{w^\bullet_{\Sigma\Sigma}} X_{jt}\,, \tag{8}$$

$$S^{2\bullet} = \frac{1}{J(T-1)} \sum_{j=1}^{J}\sum_{t=1}^{T} w^\bullet_{jt}(X_{jt} - X^\bullet_{jw})^2\,; \tag{9}$$

$$A^\bullet = \frac{w^\bullet_{\Sigma\Sigma}}{w^{\bullet 2}_{\Sigma\Sigma} - \Sigma_j w^{\bullet 2}_{j\Sigma}} \left(\sum_{j=1}^{J} w^\bullet_{j\Sigma}(X^\bullet_{jw} - X^\bullet_{ww})^2 - (J-1)S^{2\bullet} \right). \tag{10}$$

Like X_{ww} the estimator $M^\bullet$ is an unbiased estimator for m. The expected value of $S^{2\bullet}$ has been derived in De Vylder and Goovaerts (1991):

$$E[S^{2\bullet}] = \frac{s^2}{J(T-1)} \sum_{j=1}^{J}\sum_{t=1}^{T} \frac{w^{\bullet 2}_{jt}}{w_{jt}} \left(\frac{1}{w^\bullet_{jt}} - \frac{1}{w^\bullet_{j\Sigma}} \right). \tag{11}$$

By using

$$\mathrm{Cov}[X^\bullet_{jw}, X^\bullet_{kw}] = \delta_{jk} \left(a + s^2 \sum_{t=1}^{T} \frac{w^{\bullet 2}_{jt}}{w^{\bullet 2}_{j\Sigma}} \frac{1}{w_{jt}} \right), \tag{12}$$

to calculate $E[(X^\bullet_{jw} - X^\bullet_{ww})^2]$, it can be derived that the expected value of $A^\bullet$ is equal to

$$E[A^\bullet] = a + s^2 \frac{w^\bullet_{\Sigma\Sigma}}{w^{\bullet 2}_{\Sigma\Sigma} - \Sigma_j w^{\bullet 2}_{j\Sigma}} h_{\Sigma\Sigma} \tag{13}$$

with

$$h_{jt} = \frac{w^{\bullet 2}_{jt}}{w_{jt}} \left(\frac{1}{J(T-1)} \left(\frac{JT-1}{w^\bullet_{j\Sigma}} - \frac{J-1}{w^\bullet_{jt}} \right) - \frac{1}{w^\bullet_{\Sigma\Sigma}} \right). \tag{14}$$

Thus, if the estimators for s^2 and a are based on the wrong weights they are in general biased. However, an exception to this rule holds for $A^\bullet$ if the classical Bühlmann model is applied mistakenly. In that case all $w^\bullet_{jt}$ are equal to one and, consequently, the h_{jt} are zero. From this it follows that $A^\bullet$ is unbiased. Since this situation takes a central place in this article, we define the corresponding expected value of $S^{2\bullet}$ as

$$s^{2\bullet} = \frac{s^2}{JT} \sum_{j=1}^{J} \sum_{t=1}^{T} \frac{1}{w_{jt}}. \tag{15}$$

If the estimators $S^{2\bullet}$ and $A^\bullet$ are substituted for the parameters s^2 and a in the classical Bühlmann credibility premium, then an approximation for the resulting credibility factor is $z^\bullet = aT/(aT + s^{2\bullet})$. An asymptotic justification of this approximation will be given in Section 4. In the next section it is shown that this credibility factor is the optimal credibility factor which is identical for all contracts in the Bühlmann-Straub model.

3 The classical Bühlmann premium in the Bühlmann-Straub model

We assume that the observed risks are distributed according to the assumptions of the Bühlmann-Straub model, but that the credibility premium is restricted to be of the same nature as the credibility premium in the classical Bühlmann model. More specifically, the premium for a certain contract is equal to the weighted average between m and the ordinary average of the observed claims for that contract, with a credibility factor that is the same for all contracts.

If the variance parameters are known, a possible premium to choose in this situation is the theoretical Bühlmann premium in (1). However, the corresponding credibility factor $z = aT/(aT + s^2)$ is not optimal as will be shown in the first sub-section. It turns out that the optimal credibility factor is equal to the approximated credibility factor $z^\bullet = aT/(aT + s^{2\bullet})$ in the previous section. This credibility factor is generally smaller than the average of the Bühlmann-Straub credibility factors, which means that the restricted premium implies more solidarity among the insured on average. In the second sub-section it is shown how these results can be extended to an unbalanced set of observations.

3.1 The optimal classical-Bühlmann-like premium

We are looking for the optimal credibility premium in the class of premiums of the form

$$P_j^{B,\,BS} = \tilde{z}\overline{X}_j + (1 - \tilde{z})m\,. \tag{16}$$

Minimization of the total mean squared error

$$\sum_{j=1}^{J} E[\{\mu(\Theta_j) - z\overline{X}_j - (1 - z)m\}^2] = \sum_{j=1}^{J}(a - 2az + z^2 \operatorname{Var}[\overline{X}_j])\,, \tag{17}$$

with respect to z gives

$$\tilde{z} = \frac{aJ}{\sum\limits_{j=1}^{J} \operatorname{Var}[\overline{X}_j]} = \frac{aT}{aT + \frac{s^2}{JT}\sum\limits_{j=1}^{J}\sum\limits_{t=1}^{T}\frac{1}{w_{jt}}}\,. \tag{18}$$

This credibility factor is identical to the credibility factor $z^\bullet$. Therefore, if the weights in the Bühlmann-Straub model are neglected one approximately uses the optimal classical-Bühlmann-like premium.
By applying Jensen's inequality, we find

$$\begin{aligned}\tilde{z} &= \frac{aT}{aT + \frac{s^2}{J}\sum\limits_{j=1}^{J}\left(\sum\limits_{t=1}^{T}\frac{1}{T}\frac{1}{w_{jt}}\right)} \\ &\leq \frac{aT}{aT + s^2\sum\limits_{j=1}^{J}\frac{1}{w_{j\Sigma}/T}} = \frac{1}{\frac{1}{J}\sum\limits_{j=1}^{J}\frac{1}{z_j}}\,.\end{aligned} \tag{19}$$

The right hand side of this expression is equal to the credibility factor that is obtained by minimizing the following mean squared error:

$$\sum_{j=1}^{J} E[\{\mu(\Theta_j) - zX_{jw} - (1-z)m\}^2] \,. \tag{20}$$

So, lower credibility is given to the ordinary average $\overline{X}_j$ than to the weighted average X_{jw}. This can be explained by the fact that X_{jw} contains more precise information about $\mu(\Theta_j)$ than $\overline{X}_j$ does, in the sense that $E[\{\mu(\Theta_j) - X_{jw}\}^2] \leq E[\{\mu(\Theta_j) - \overline{X}_j\}^2]$. Furthermore, with Jensen's inequality it follows that

$$\left(\frac{1}{J}\sum_{j=1}^{J}\frac{1}{z_j}\right)^{-1} \leq z_\Sigma / J \,. \tag{21}$$

This means that, besides the replacement of X_{jw} with $\overline{X}_j$, the restriction of an identical credibility factor also makes a contribution to the fact that the premiums based on $\tilde{z}$ attach less weight to the individual experience with the contracts than the Bühlmann-Straub credibility premiums do. Consequently, with the optimal classical-Bühlmann-like premiums the insured are more solidary in their premiums. This conclusion is supported by the comparison of the variances of the total premium income:

$$\mathrm{Var}\left[\sum_{j=1}^{J} P_j^{BS}\right] = \sum_{j=1}^{J} \mathrm{Var}\left[P_j^{BS}\right] = a z_\Sigma \,, \tag{22}$$

$$\mathrm{Var}\left[\sum_{j=1}^{J} P_j^{B,\,BS}\right] = \sum_{j=1}^{J} \mathrm{Var}[P_j^{B,\,BS}] = aJ\tilde{z} \,. \tag{23}$$

The inequalities in (19) and (21) imply that the variance of the total premium income is smaller for the restricted premiums. The Bühlmann-Straub premium allows for a better differentiation of the premiums and therefore implies a higher variance of the total premium income. Stabilization of the total premium can therefore be an argument to apply the restricted credibility estimators. One should, however, note that the ultimate stabilization is obtained by taking each premium equal to m. It is then needless to use a credibility approach.

Finally, if the weights in the Bühlmann-Straub model are identical for all risks in each contract (i.e. $w_{jt} = w_j$ for all t), then (19) becomes an equality and the relation between $\tilde{\tilde{z}}$ and the z_j is given by

$$\frac{1}{\tilde{\tilde{z}}} = \frac{1}{J}\sum_{j=1}^{J}\frac{1}{z_j}. \tag{24}$$

In all other cases this equality is replaced with a greater than inequality.

3.2 *Results for an unbalanced set of observations*

In an unbalanced set of observations, the number of observed risks differs from contract to contract. For contract j we define this number as T_j, so that the total number of observations is equal to T_Σ. The estimator we consider for a remains of the form in (7), but for s^2 we base our results on

$$S^2 = \frac{1}{J}\sum_{j=1}^{J} S_j^2. \tag{25}$$

Here, S_j^2 is the following unbiased estimator for s^2 based on only the observations for contract j:

$$S_j^2 = \frac{1}{T_j - 1}\sum_{t=1}^{T_j} w_{jt}(X_{jt} - X_{jw})^2. \tag{26}$$

For the existence of these statistics, we assume that the T_j are greater than one. Another estimator for s^2 can for instance be found by replacing $J(T-1)$ in (6) with $T_\Sigma - J$, but that estimator is not suitable for our analysis.
If $S_j^{2\bullet}$ is defined as S_j^2 based on incorrect weights, then its expected value is equal to

$$E[S_j^{2\bullet}] = \frac{s^2}{T_j - 1}\sum_{t=1}^{T_j}\frac{w_{jt}^{\bullet 2}}{w_{jt}}\left(\frac{1}{w_{jt}^{\bullet}} - \frac{1}{w_{j\Sigma}^{\bullet}}\right). \tag{27}$$

The expected value of $A^{\bullet}$ is given by (13) and (14), but with T replaced by T_j. Again, $A^{\bullet}$ remains unbiased if the classical Bühlmann is applied mistakenly. In that case we denote the expected value of $S_j^{2\bullet}$ by

$$s_j^{2\bullet} = \frac{1}{T_j} \sum_{t=1}^{T_j} \frac{1}{w_{jt}} . \tag{28}$$

If the assumptions of the classical Bühlmann model hold and the set of observations is unbalanced then the credibility factors also differ from contract to contract, just as in the Bühlmann-Straub model. The optimal credibility factor that is identical for all contracts in the classical Bühlmann model is then equal to

$$z = \frac{a}{a + \frac{1}{J} \sum_{j=1}^{J} s^2/T_j} . \tag{29}$$

This value can be found by minimizing of the total mean squared error given in (17), with the assumptions (i)–(iv). To make this credibility premium operational, we replace a with A and, for each value of the index j in the occurring sum, s^2 with the contract-based estimator S_j^2:

$$Z = \frac{A}{A + \frac{1}{J} \sum_{j=1}^{J} S_j^2/T_j} . \tag{30}$$

If the model turns out to be the Bühlmann-Straub model instead of the classical Bühlmann model the credibility factor is approximately equal to

$$z^{\bullet} = \frac{a}{a + \frac{1}{J} \sum_{j=1}^{J} s_j^{2\bullet}/T_j} = \frac{a}{a + \frac{s^2}{J} \sum_{j=1}^{J} \frac{1}{T_j^2} \sum_{t=1}^{T_j} \frac{1}{w_{jt}}} . \tag{31}$$

This expression is obtained by substituting the expected values of $A^{\bullet}$ and $S_j^{2\bullet}$ for A and S_j^2 in (30). The value of $z^{\bullet}$ is again equal to the credibility factor in the optimal classical-Bühlmann-like premium, which can be derived similarly to the credibility factor in (18). The other results given in the previous sub-section also hold for the credibility factor in (31). This can be shown by using the same methods.

4 Asymptotical results

In the previous section it has been shown that neglecting the weights in the Bühlmann-Straub model boils down to approximately using the optimal premium in the class of premiums satisfying equation (16). If the number of contracts gets larger, then it may be expected that this approximation tends to be closer. In this section it will be shown that this statement holds indeed.

For that purpose we will assume that the design underlying the risks is random. This has also been done in other asymptotic studies in credibility theory (e.g. Neuhaus, 1984, and Hesselager, 1988). In the first place this means that we assume that the weights are realizations of independently and identically distributed random variables W_{jt}, with $E[1/W_{jt}] = g$. The credibility premiums are obtained by minimization of the mean squared error conditionally to the realizations of the W_{jt}. If this condition is dropped, a classical Bühlmann model results in which the within-variance is equal to gs^2.

In order to derive asymptotical results for unbalanced data sets, we also assume that the numbers of observed risks T_j are independently and identically distributed random variables, with finite second moments and $E[T_j]$ defined as f and $E[1/T_j]$ as h. The probability limit for $J \to \infty$ of $Z^\bullet$ in (31) is then

$$\begin{aligned}
\operatorname{plim} Z^\bullet &= \frac{a}{a + s^2 \operatorname{plim}\left(\frac{1}{J}\sum_{j=1}^{J}\frac{1}{T_j^2}\sum_{t=1}^{T_j}\frac{1}{W_{jt}}\right)} \\
&= \frac{a}{a + s^2 E\left(\frac{1}{T_1}\frac{1}{T_1}\sum_{t=1}^{T_1}\frac{1}{W_{1t}}\right)} \\
&= \frac{a}{a + s^2 gh}\,. \qquad (32)
\end{aligned}$$

Here we have used Slutsky's theorem and the weak law of large numbers. Furthermore, we have:

$$\operatorname{plim} \frac{1}{J}\sum_{j=1}^{J} \frac{S_j^{2\bullet}}{T_j} = E\left[E\left[\sum_{t=1}^{T_1}(X_{1t}-\overline{X}_1)^2/(T_1(T_1-1)) \mid T_1, W_{11}, \ldots, W_{1T_1}\right]\right]$$

$$= s^2 gh\,, \tag{33}$$

and, with $\overline{X} = \Sigma_j \overline{X}_j / J$:

$$\operatorname{plim} A^\bullet = \operatorname{plim}\left(\frac{\frac{T_\Sigma}{J}}{\frac{T_\Sigma^2}{J^2} - \frac{1}{J}\left(\frac{1}{J}\sum_{j=1}^{J} T_j^2\right)}\right)$$

$$\cdot \operatorname{plim}\left(\frac{1}{J}\sum_{j=1}^{J} T_j(\overline{X}_j - \overline{X})^2 - \left(1-\frac{1}{J}\right)S^{2\bullet}\right)$$

$$= \frac{f}{f^2 - 0}\cdot\left(\operatorname{plim}\frac{1}{J}\sum_{j=1}^{J} T_j(\overline{X}_j - m)^2 - \operatorname{plim} S^{2\bullet}\right)$$

$$= \left(E[T_1 \operatorname{Var}[\overline{X}_1 \mid T_1, W_{11}, \ldots, W_{1T_1}]] - s^2 g\right)/f$$

$$= (fa + s^2 g - s^2 g)/f = a\,.$$

Consequently,

$$\operatorname{plim} \frac{A^\bullet}{A^\bullet + \frac{1}{J}\sum_{j=1}^{J} S_j^{2\bullet}/T_j} = \frac{a}{a + s^2 gh} = \operatorname{plim} Z^\bullet \tag{35}$$

This means that when the weights in the Bühlmann-Straub model are neglected, the used credibility factor tends to the credibility factor in the optimal classical-Bühlmann-like premium if the number of contracts goes to infinity. The fact that the restricted credibility factor is less than or equal to the average of the Bühlmann-Straub credibility factors also holds for their probability limits.

5 Numerical example

The results in this paper will be illustrated in this section by an application to collective automobile insurance data. We consider a portfolio consisting of nine fleets of cars for which ten years of observation are available. Table 2 in the Appendix contains the average claim figures per car, expressed in an unknown currency. In Table 3 the corresponding numbers of cars are tabulated. The total risk exposure in the portfolio is 1510 years.
The realizations of the estimators M, S^2 and A in the Bühlmann-Straub model are

$$m^* = 489.83\,, \quad s^{2*} = 695107.00\,, \quad a^* = 26195.97\,. \tag{36}$$

In the second and third row of Table 1 below the estimated credibility factors and the credibility premiums are given. The average of the Bühlmann-Straub credibility factors is equal to 0.791. The mean squared error for the premium per car in fleet j is equal to $a(1 - z_j)$. As a measure for the goodness of fit of the Bühlmann-Straub model we take the sum of these mean squared errors that is estimated to be 49322.

Table 1 Estimated credibility factors and credibility premiums.

j	1	2	3	4	5	6	7	8	9
z_j	0.952	0.904	0.693	0.839	0.868	0.601	0.856	0.828	0.576
P_j^{BS}	506	203	343	373	626	282	441	495	644
$P_j^{B,\,BS}$	476	272	321	411	551	300	442	461	566

If we assume that the numbers of cars in Table 3 are not available for some reason and the classical Bühlmann model is used, then the estimates for the parameters obtained from the statistics $M^\bullet$, $S^{2\bullet}$ and $A^\bullet$ in (8), (9) and (10) are

$$m^{\bullet *} = 422.21\,, \quad s^{2\bullet *} = 112784.24\,, \quad a^{\bullet *} = 18203.19\,. \tag{37}$$

The estimates for the within- and between-variance are both lower than in (36), but the decrease of the within-variance is largest. The estimated credibility factor is 0.617, which is lower than the average of the Bühlmann-Straub credibility factors indeed. Adding the premiums in the last row of

Table 1 gives the total amount of 3800. This is 2.9 % lower than the sum of the premiums in the Bühlmann-Straub model. Table 1 also shows that the variability of the premiums $P_j^{B,BS}$ is lower than the variance of the Bühlmann-Straub premiums P_j^{BS}.
If the estimates s^{2*} and a^* are substituted for s^2 and a in the expressions (11) and (13) for the expected values of $S^{2\bullet}$ and $A^\bullet$, we obtain

$$E[S^{2\bullet}]^* = \frac{s^{2*}}{JT} \sum_{j=1}^{J} \sum_{t=1}^{T} \frac{1}{w_{jt}} = 97373.54\,; \qquad (38)$$

$$E[A^\bullet]^* = a^* = 26195.97\,. \qquad (39)$$

We can see that the difference between $s^{2\bullet*}$ and $E[S^{2\bullet}]^*$ is about 16 %, while this is 44 % between $a^{\bullet*}$ and $E[A^\bullet]^*$. The estimated optimal classical-Bühlmann-like credibility factor $\tilde{z}$ is 0.735. This value is 19 % greater than the credibility factor that resulted in the classical Bühlmann model, which implies that the used approximation is a bit crude for this small portfolio of 9 contracts.
The mean squared error of the sum of the optimal classical-Bühlmann-like premium is equal to $Ja(1 - \tilde{z}^{BL})$ that is estimated to be 62441. We take this value as an approximation for the total mean squared error of the Bühlmann premium in the Bühlmann-Straub model. Neglecting the weights then leads to an increase of the total mean squared error of about 26.5 % compared to the value of 49322 corresponding to the Bühlmann-Straub premiums.

6 Conclusions

In the Bühlmann-Straub model with known parameters the classical Bühlmann premium is not the optimal credibility premium in the class of all 'classical-Bühlmann-like' premiums. The optimal premium in this class is based on a credibility factor that is smaller than the average of the Bühlmann- Straub credibility factors. This implies that the insured are more solidary in their premiums and that the total premium income is more stable.
If incorrect weights are used in the Bühlmann-Straub model, then the estimators for the parameters measuring the within- and between-variance are

generally biased. An exception to this rule holds if the classical Bühlmann model is applied mistakenly, which is the central case of this study. Then the estimator for the between-variance is unbiased, although the estimator concerning the within-variance is still biased. Substitution of the expected value of the latter estimator in the formula for Bühlmann's credibility estimator, however, shows that approximately the optimal classical-Bühlmann-like estimator is used. This approximation is shown to be exact if the number of contracts in the insurance portfolio tends to infinity: The probability limit of the credibility factor is equal to the credibility factor in the optimal classical-Bühlmann-like premium.

References

Bühlmann, H. (1967). Experience Rating and Credibility; ASTIN Bulletin, 199–207.

Bühlmann, H. (1969). Experience Rating and Credibility; ASTIN Bulletin, 157–165.

Bühlmann, H., and E. Straub (1970). Glaubwürdigkeit für Schadensätze; Mitteilungen der Vereinigung schweizerischer Versicherungsmathematiker, 111–133.

Goovaerts, M.J., R. Kaas, A.E. van Heerwaarden and T. Bauwelinckx (1990). Effective Actuarial Methods, North-Holland, Amsterdam.

Hesselager, O. (1988). On the Asymptotic Distribution of Weighted Least Squares Estimators; Scandinavian Actuarial Journal, 69–76.

Neuhaus, W. (1984). Inference about Parameters in Empirical Linear Bayes Estimation Problems; Scandinavian Actuarial Journal, 131–142.

Vylder, F. de (1981). Practical Credibility Theory with Emphasis on Optimal Parameter Estimation; ASTIN Bulletin, 115–131.

Vylder, F. de, and M.J. Goovaerts (1991). Estimation of the Heterogeneity Parameter in the Bühlmann-Straub Credibility Theory Model; Insurance: Mathematics and Economics, 233–238.

D.R. Dannenburg
Department of Actuarial Science & Econometrics
University of Amsterdam
Roetersstraat 11
NL –1018 WB Amsterdam

Appendix: Data used in the numerical example

Table 2 Average claims in nine fleets of cars during ten years.

year	1	2	3	4	5	6	7	8	9	10
fleet 1	540	514	576	483	481	493	438	588	541	441
fleet 2	99	103	163	126	0	219	370	273	155	275
fleet 3	0	400	1042	313	0	833	0	0	0	0
fleet 4	275	278	430	196	667	185	517	204	323	968
fleet 5	543	984	727	562	722	610	794	299	580	488
fleet 6	0	0	0	645	833	0	0	769	0	0
fleet 7	333	404	400	361	588	349	435	476	635	556
fleet 8	494	133	735	519	1000	641	339	513	227	244
fleet 9	1667	313	556	769	1818	0	1429	0	0	0

Table 3 Numbers of insured cars.

year	1	2	3	4	5	6	7	8	9	10
fleet 1	44	50	56	58	58	56	54	52	52	46
fleet 2	20	20	24	32	28	28	28	22	26	22
fleet 3	8	6	10	6	8	4	6	4	4	4
fleet 4	22	22	18	20	12	10	12	10	6	6
fleet 5	26	24	22	18	20	16	12	14	14	8
fleet 6	6	8	6	6	2	4	2	2	2	2
fleet 7	18	20	20	16	18	18	14	12	12	10
fleet 8	16	16	14	16	14	16	12	8	8	8
fleet 9	6	6	4	2	4	2	4	2	4	2

Summary

The properties of the classical Bühlmann premium are considered in case it is used in the more general Bühlmann-Straub model. Reasons for not applying the Bühlmann-Straub premiums may be to establish more solidarity among the insured or the fact that no proper measures for the weights in the Bühlmann-Straub model are available. The optimal 'classical-Bühlmann-like' premium is derived if the parameters in the Bühlmann-Straub model are known. For cases in which these parameters are not known, some estimators are examined if they are based on the incorrect assumptions of the classical Bühlmann model. It turns out that substitution of these estimators into the classical Bühlmann premium approximately leads to the optimal classical-Bühlmann-like premium in the Bühlmann-Straub model with known parameters. A numerical example based on collective automobile insurance data illustrates the results.

Zusammenfassung

Die Eigenschaften der klassischen Bühlmann-Prämie werden untersucht, für den Fall, dass sie im Rahmen des allgemeineren Bühlmann-Straub-Modells eingesetzt wird. Gründe, um die Bühlmann-Straub-Prämie nicht anzuwenden, können darin liegen, dass mehr Solidarität unter den Versicherten angestrebt wird, oder in der Tatsache, dass keine geeigneten Masse für die Gewichte im Bühlmann-Straub-Modell vorliegen. Es wird die optimale "klassisch-Bühlmann-artige" Prämie hergeleitet, wenn die Parameter im Bühlmann-Straub-Modell bestimmbar sind. Für Fälle, wo diese Parameter unbekannt sind, werden einige Schätzer untersucht, für welche die hier falschen Annahmen des klassischen Bühlmann-Modells zugrunde gelegt werden. Es zeigt sich, dass die Substitution dieser Schätzer in die klassische Bühlmann-Prämie approximativ zur optimalen klassisch-Bühlmann-artigen Prämie im Bühlmann-Straub-Modell mit bekannten Parametern führt. Ein numerisches Beispiel anhand von Daten aus der Motorfahrzeug-Flotten-Versicherung illustriert diese Resultate.

Résumé

On considère les propriétés de la prime du modèle classique de Bühlmann dans le cas où elle est utilisée dans le modèle plus général de Bühlmann-Straub. Les raisons pour lesquelles la prime n'est pas déterminée en appliquant le modèle de Bühlmann-Straub peuvent être d'augmenter la solidarité entre assurés ou le fait qu'aucune mesure adéquate des poids du modèle de Bühlmann-Straub n'est connue. L'équivalent optimal de la prime du modèle classique de Bühlmann est déterminé dans les cas où les paramètres du modèle de Bühlmann-Straub sont connus. Dans le cas où ces paramètres ne sont pas connus on étudie quelques estimateurs, partant d'hypothèses incorrectes du modèle classique de Bühlmann. Il s'ensuit que la substitution de ces estimateurs dans la prime du modèle classique de Bühlmann donne une approximation de l'équivalent optimal de la prime du modèle classique de Bühlmann dans le modèle de Bühlmann-Straub avec des parametrès connus. On illustre ces résultats à l'aide d'un exemple provenant de l'assurance de flottes automobiles.