Kurzmitteilungen

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An upper limit of the expected shortfall

1 Introduction

For many years reinsurers loaded risk premiums with loadings proportional to the expected value, to the variance or to the standard deviation of the aggregate claims. Recently, some reinsurers have replaced such loadings by a loading proportional to the difference between the expected shortfall and the expected value of the aggregate claims. The simple upper limit of the expected shortfall shown in this paper may be helpful for the actuarial practitioner. If the distribution function of the aggregate claims is continuous the result is the same as theorem 2 in Bertsimas et al. (2004). For the discontinuous case, on the other hand, the author did not come across any reference in the actuarial literature.

2 Deduction of the upper limit

Let $X \geq 0$ be a random variable, $F(x) = \text{Prob}(X \leq x)$ the distribution function of X, $E = E[X]$ and $V = \text{Var}(X)$. Consider an arbitrary probability p in the interval $0 < p < 1$ and let u denote an x-value for which $F(u) = p$. Such an u is called a p-quantile of the distribution of X. Note that $F(x)$ is assumed to be continuous at u. The conditional expected value $E[X \mid X > u]$ is called expected shortfall. Typically, p is chosen as 99%. The expected values of X and of X^2 can be split into

$$E[X] = p \cdot E[X \mid X \leq u] + (1-p) \cdot E[X \mid X > u] \tag{1}$$

and

$$E[X^2] = p \cdot E[X^2 \mid X \leq u] + (1-p) \cdot E[X^2 \mid X > u]\,. \tag{2}$$

Using the abbreviations

$$a = E[X \mid X \leq u] \quad \text{and} \quad b = E[X \mid X > u]\,, \tag{3}$$

the conditional expected values to the right of (2) can be written as

$$E[X^2 \mid X \leq u] = a^2 + \text{Var}(X \mid X \leq u)$$

and

$$E[X^2 \mid X > u] = b^2 + \mathrm{Var}(X \mid X > u)\,,$$

hence, since the variances are ≥ 0,

$$E[X^2] \geq p \cdot a^2 + (1-p) \cdot b^2. \tag{4}$$

Consider the two-point distribution which assumes the two values a and b with the probabilities p and $1-p$, respectively. It follows from (1) and (3) that its expected value is E. Let W stand for the variance of this two-point distribution. From (4) it is seen that

$$V \geq W\,. \tag{5}$$

For two-point distributions with expectation E and variance W there are the relations

$$p \cdot a + (1-p) \cdot b = E$$

and

$$p \cdot a^2 + (1-p) \cdot b^2 = E^2 + W\,.$$

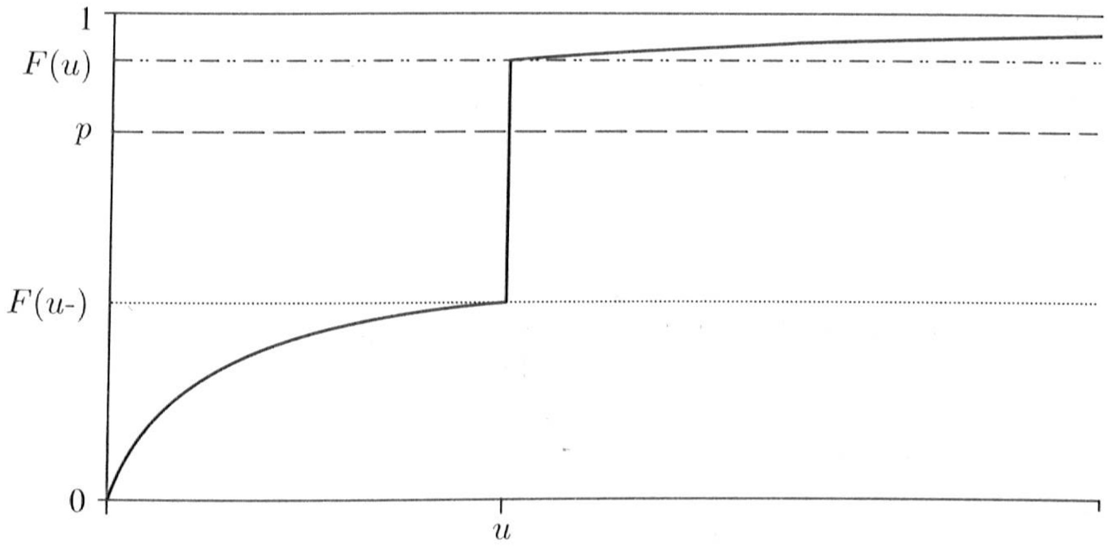
Solving for a and b we obtain

$$a = E - \sqrt{\frac{W \cdot (1-p)}{p}} \quad \text{and} \quad b = E + \sqrt{\frac{W \cdot p}{1-p}}\,. \tag{6}$$

The upper limit of the expected shortfall is now found by

$$\begin{aligned} E[X \mid X > u] &= b && \text{(using (3))} \\ &= E + \sqrt{\frac{W \cdot p}{1-p}} && \text{(using (6))} \\ &\leq E + \sqrt{\frac{V \cdot p}{1-p}} && \text{(using (5))}\,. \end{aligned} \tag{7}$$

If $F(x)$ is not continuous for all $x \geq 0$ there may be no p-quantile but $F(u) > p$ and $F(u\text{-}) < p$ ($F(u\text{-})$ being the limit of $F(x)$ for $x < u$ when $x \to u$). Such a situation is shown in the following graph.



In this case we have

$$\begin{aligned} E[X \mid X < u] &= \int_0^u \left(1 - \frac{F(x)}{F(u\text{-})}\right) dx \\ E[X \mid X > u] &= u + \int_u^\infty \left(1 - \frac{F(x) - F(u)}{1 - F(u)}\right) dx \end{aligned} \tag{8}$$

and

$$\begin{aligned} E = F(u\text{-}) \cdot E[X \mid X < u] &+ \big((F(u) - F(u\text{-})\big) \cdot u \\ &+ \big(1 - F(u)\big) \cdot E[X \mid X > u]\,. \end{aligned}$$

The expected shortfall as defined so far, $E[X \mid X > u]$ is now slightly modified, namely to

$$\frac{E[X \mid X > u] \cdot \big(1 - F(u)\big) + u \cdot \big(F(u) - p\big)}{1 - p}\,. \tag{9}$$

The numerator in (9) is equal to the shaded area in the graph. In the appendix it is shown that (9) is equivalent to the definition given in Acerbi and Tasche (2002).

In order to deduce the upper limit of this modified expected shortfall, the two conditional random variables $X \mid X \leq u$ and $X \mid X > u$ are replaced by X_1 and

X_2 defined by their distribution functions:

$$G_1(x) = \text{Prob}(X_1 \leq x) = \begin{cases} \dfrac{F(x)}{p} & \text{for } 0 \leq x < u \\ 1 & \text{for } x \geq u \end{cases}$$

$$G_2(x) = \text{Prob}(X_2 \leq x) = \begin{cases} 0 & \text{for } x < u \\ 1 - \dfrac{1 - F(x)}{1 - p} & \text{for } x \geq u \end{cases}$$

It follows from these definitions that $F(x)$, the distribution function of X, is the mixture $p \cdot G_1(x) + (1-p) \cdot G_2(x)$.
The expected value of X_1 is

$$\begin{aligned} E[X_1] &= \int_0^u \left(1 - G_1(x)\right) dx \\ &= \int_0^u \left(1 - \frac{(F(x)}{p}\right) dx \\ &= \frac{F(u\text{-})}{p} \cdot \int_0^u \left(\frac{p}{F(u\text{-})} - 1 + 1 - \frac{F(x)}{F(u\text{-})}\right) dx \\ &= \frac{F(u\text{-})}{p} \cdot \left(\frac{p - F(u\text{-})}{F(u\text{-})} \cdot u + E[X \mid X < u]\right) \qquad \text{(using (8).)} \end{aligned}$$

In a similar way the expected value of X_2 is

$$\begin{aligned} E[X_2] &= \int_0^\infty \left(1 - G_2(x)\right) dx \\ &= \int_0^u dx + \int_u^\infty \frac{1 - F(x)}{1 - p} dx \\ &= u + \frac{1 - F(u)}{1 - p} \cdot \int_u^\infty \frac{1 - F(u) + F(u) - F(x)}{1 - F(u)} dx \\ &= u \cdot \frac{F(u) - p}{1 - p} + \frac{1 - F(u)}{1 - p} \cdot E[X \mid X > u] \qquad \text{(using (8).)} \qquad (10) \end{aligned}$$

From (9) and (10) it is seen that the expected shortfall is $E[X_2]$. Replacing the conditional random variables $X \mid X \leq u$ and $X \mid X > u$ by X_1 and X_2, respectively, in (1), (2), (3), (4) and (7) we obtain the same upper bound of the expected shortfall as in the continuous case.

3 Numerical examples

The truncated Pareto distribution which is often used to model the distribution of excess losses may serve as an example of a continuous distribution. The necessary algebraic formulas are summarised for the cover c after the deductible d (the layer $c\ xs\ d$ in the usual reinsurance notation):

$$F(x) = \begin{cases} 1 - \left(\dfrac{d}{d+x}\right)^{\alpha} & \text{if } 0 \leq x < c \\ 1 & \text{if } x \geq c \end{cases} \tag{11}$$

Let $M = d + c$ and $k = \dfrac{M}{d}$. Then

$$E[X] = d \cdot \frac{1 - k^{1-\alpha}}{\alpha - 1} \tag{12}$$

$$E[X \mid X > u] = u + (d+u) \cdot \frac{1 - \left(\dfrac{M}{d+u}\right)^{1-\alpha}}{\alpha - 1} \tag{13}$$

$$\mathrm{Var}(X) = 2 \cdot d^2 \cdot \left(\frac{k^{1-\alpha} - 1}{\alpha - 1} - \frac{k^{2-\alpha} - 1}{\alpha - 2}\right) - E[X]^2 \tag{14}$$

Choose for the numerical example $\alpha = 1.2$, $d = 1$, $c = 19$ and let $p = 0.95$. From (11) we obtain the solution $u = 11.139$ of the equation $F(u) = 0.95$ and from (13) the expected shortfall $E[X \mid X > u] = 16.907$. Using (12) and (14) we have $E[X] = 2.254$ and $\mathrm{Var}(X) = 15.378$, and with (7) the upper limit of the expected shortfall becomes 19.347.

In order to illustrate the non-continuous case we take the Poisson distribution with $\lambda = 0.2$ and choose $p = 0.99$. For $u = 2$ we find $F(u\text{-}) = 0.9825$ and

$F(u) = 0.9989.$

$$\begin{aligned}E[X \mid X > u] \cdot (1 - F(u)) &= \sum_{n=u+1}^{\infty} e^{-\lambda} \cdot \frac{\lambda^n}{n!} \cdot n \\ &= \sum_{n=0}^{\infty} e^{-\lambda} \cdot \frac{\lambda^n}{n!} \cdot n - \sum_{n=0}^{u} e^{-\lambda} \cdot \frac{\lambda^n}{n!} \cdot n \\ &= \lambda - \lambda \cdot \sum_{n=1}^{u} e^{-\lambda} \cdot \frac{\lambda^{n-1}}{(n-1)!} \\ &= 0.2 - 0.2 \cdot 0.9825 \\ &= 0.0035\end{aligned}$$

According to (9) the expected shortfall is thus 2.1. On the other hand, the upper limit is equal to

$$0.2 + \sqrt{\frac{0.2 \cdot 0.99}{0.01}} = 4.6\,.$$

References

Acerbi, C., Tasche, D. (2002) Expected Shortfall: a natural coherent alternative to Value at Risk. Economic Notes by Banca Monte dei Paschi di Siena SpA, vol. 31, no. 2, pp. 379–388.

Bertsimas, D., Lauprete, G.J., Samarov, A. (2004) Shortfall as a risk measure: properties, optimization and applications. Journal of Economic Dynamics and Control, vol. 28, no. 7, pp. 1353–1381.

Appendix

The random variable X in Acerbi and Tasche (2002) is replaced by Y in order to distinguish it from the random variable X in the present paper. Then the following notation is used:
Y is a profit-loss random variable, α is a probability, $y^{(\alpha)} = \sup\{y \mid P[Y \le y] \le \alpha\}$ is a quantile of the distribution of Y. The expected shortfall is defined as

$$ES^{(\alpha)}(Y) = -\frac{1}{a}\Big(E\big[Y\,\mathbf{1}_{\{Y \le y^{(\alpha)}\}}\big] - y^{(\alpha)}\big(P\big[Y \le y^{(\alpha)}\big] - \alpha\big)\Big)\,. \tag{15}$$

Using $X = -Y$, $x = -y$, $u = -y^{(\alpha)}$ and $p = 1 - \alpha$ we have

$$\mathrm{Prob}(Y \le y) = 1 - \mathrm{Prob}(X < x)$$

and therefore

$$\begin{aligned} P[Y \le y^{(\alpha)}] - \alpha &= 1 - \mathrm{Prob}(X < u) - 1 + p \\ &= p - F(u\text{-})\,. \end{aligned} \tag{16}$$

Together with

$$\begin{aligned} E\big[Y\,\mathbf{1}_{\{Y \le y^{(\alpha)}\}}\big] &= -E\big[X\,\mathbf{1}_{\{X \ge u\}}\big] \\ &= -E[X \mid X > u]\big(1 - F(u)\big) - u\big(F(u) - F(u\text{-})\big) \end{aligned} \tag{17}$$

the expression in the bracket to the right of (15) is

$$-E[X \mid X > u]\big(1 - F(u)\big) - u\big(F(u) - p\big)\,.$$

After multiplication with $\dfrac{-1}{1-p}$ this is equal to (9).

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