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The equilibrium points of Lagrange

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Résumé: Grâce à Lagrange on sait depuis 1770 que lorsqu'une planète est en orbite circulaire autour du Soleil, il y a cinq points liés à ce système où, sous certaines conditions, des objets de très petite masse peuvent se trouver piégés. On parle aujourd'hui de points d'équilibre de Lagrange. En réalité seul les points L_4 et L_5, qui sont 60° en avance et 60° en retard sur l'orbite de la planète, ont vraiment cette propriété. Ce qui d'abord surprend, est que ces points sont des maxima du potentiel généralisé calculé dans le système de coordonnées en rotation avec la planète. Si un objet de petite masse se trouve à l'un de ces points, il cherchera naturellement à s'en éloigner pour aller vers des régions de plus faible potentiel; mais il sera maintenu dans le voisinage du point par la force de Coriolis, cette force que l'on perçoit, en plus de la force centrifuge, lorsqu'on essaye de se déplacer radialement sur un carrousel qui tourne. Les autres points, L_1 à L_3, sont tous sur l'axe reliant les deux corps majeurs, soit Soleil et planète ou planète et satellite. Ces points sont des points selles du potentiel et de petits objets, tels que des astéroïdes, ne peuvent pas y être piégés; mais pour les sondes spatiales il suffit de petites fusées d'appoint pour les y maintenir pendant de nombreuses années.

Zusammenfassung: Seit etwa 1770 weiss man dank Lagrange, dass, wenn ein Planet um die Sonne kreist, es fünf an diesem System gebundene Punkte gibt, wo unter gewissen Bedingungen Objekte sehr kleiner Masse eingefangen werden können. Man spricht heute von Lagrange- Gleichgewichtspunkten. In Wirklichkeit haben nur die Punkte L_4 und L_5, die auf der Bahn 60° vor und 60° hinter dem Planeten laufen, diese Eigenschaft. Was zuerst erstaunen mag, ist die Tatsache, dass an diesen Punkten das im rotierenden Planetensystem berechnete verallgemeinerte Potential maximal ist. Weilt nun ein kleines Objekt an einem solchen Punkt, dann will es sich natürlich in Richtung kleinerer Potentialwerte bewegen, wird dann aber durch die Corioliskraft (die Kraft, die man zusätzlich zur Zentrifugalkraft fühlt, wenn man sich auf einer rotierenden Drehscheibe radial bewegen will) in der Umgebung des Punktes gehalten. Die drei ersten Punkte, L_1 bis L_3, sind alle auf der Achse, zwischen den zwei Hauptkörpern, also auf der Geraden zwischen Sonne und Planet, oder entsprechend zwischen einem Planeten und einen seiner Satelliten. Diese Punkte sind Sattelpunkte des Potentials, und kleine Objekte wie Asteroide können sie nicht einfangen. Bei Raumsonden genügen jedoch kleine Bordraketen, um diese in der Umgebung von L_1 bis L_3 über mehrere Jahre zu halten.

1. Introduction

In a recent paper of this review [1] some of the numerous gravitational resonances that occur in the solar system have been described. Among these some of the most remarkable are the resonances among the Trojan satellites that revolve around the Sun near the orbit of Jupiter, as they librate around the L_4 and L_5 Lagrange equilibrium points of that planet's path.

As is well known, the three-body problem can probably not be resolved analytically. But the somewhat simpler case of two heavy objects moving on circular orbits, while the third has negligible mass, has been studied by numerous approximate methods. The analysis of this problem appears in some modern textbooks on classical mechanics, a good example being that of Fowler and Cassiday [2]. Lagrange showed in 1770 [3] that under these circumstances there are five singular points, labelled L_1 to L_5, where the gradient of the potential function locally vanishes. L_1 to L_3 are saddle points of the potential, whereas L_4 and L_5 are maxima. The surprising thing is that under special conditions, very small bodies can be trapped in the vicinity of L_4 and L_5, and these two points then become *libration centres*. Graph (a) of Fig. 1 refers to the situation where the two bodies have a mass ratio of 5 to 1, and in graph (b) that ratio is 100, close to 81, the value for the Earth-Moon system. The first three points are all located on the straight line passing through the two main objects and are often referred to as *co-linaer points*. L_4 and L_5 on the other hand, are leading and trailing the lighter of the main bodies by 60° on its orbit around the heavier one. These last two points and the two main bodies thus form a pair of equilateral triangles with a common base and are therefore often called *equilateral points*. As will be discussed later, stable libration in the vicinity of L_4 or L_5 is possible only if the mass ratio of the two large bodies exceeds **24. 96**.

Assume now that the larger body, for example a planet, has mass M_1, while a secondary body at a distance a, for example a satellite, has the smaller mass M_2. We introduce the mass ratio $q = M_1 / M_2 > 1$ and place ourselves in the co-ordinate frame where the two main bodies are at rest, as shown in Fig. 2. The potential at point $\mathbf{P}(xa, ya)$ involves the gravitational potentials of M_1 and M_2, but we must also take care of the fact that in the rotating reference frame chosen the small mass at **P** will experience a centrifugal force. This effective potential therefore comprises three terms:

$$V = -\frac{GM_1}{a \cdot \sqrt{(x + \frac{1}{q+1})^2 + y^2}} - \frac{GM_2}{a \cdot \sqrt{(x - \frac{q}{q+1})^2 + y^2}} - \frac{G(M_1 + M_2) \cdot (x^2 + y^2)}{2 \cdot a} \quad (1)$$

The first two terms represent the gravitational potentials of the main bodies, whereas the third describes the centrifugal force experienced by the small mass at P as it is rotating around the barycentre. This takes the form

$$-\frac{1}{2}\omega^2 r^2 \text{, where } r = \left(x^2 a^2 + y^2 a^2\right)^{1/2} \text{,} \quad (2)$$

and with Kepler's law,

$$\omega^2 = \frac{G(M_1 + M_2)}{a^3} \quad . \quad (3)$$

Now let

$$W = \frac{a \cdot V}{G(M_1 + M_2)} \quad . \quad (4)$$

With a little algebra we then get

$$W = -\frac{q}{\left[(1 + x(q+1))^2 + (y(q+1))^2\right]^{1/2}} - \frac{q}{\left[(q - x(q+1))^2 + (y(q+1))^2\right]^{1/2}} - \frac{1}{2} \cdot \left(x^2 + y^2\right) \quad . \quad (5)$$

W is the dimensionless ratio of the potential V to

$$\frac{G(M_1 + M_2)}{a} \quad .$$

2. Position of the equilibrium points

The locations of the five equilibrium points are obtained from the zeros of the first derivatives of W given by Eq. (5) above. We shall not give details of this calculation which can, in fact, be done in several different ways, but restrict ourselves to writing the results. The equilateral points have co-ordinates given exactly by the corresponding formulae of Table 1. The co-linear points, on the other hand, cannot be given in a closed analytical form, but must be derived numerically from equations of a higher degree. To lowest order in the mass ratio q of the two major bodies, the co-linear co-ordinates can be approximated with the formulae also given in Table 1. It is seen that these approximations are almost perfect for L_3 when $q \geq 5$. They are not as good for L_2 and quite poor for L_1, especially at low values of the mass ratio.

Fig. 2 illustrates the co-ordinate system used for the case of a mass ratio $q = 5$. It gives the positions of the major masses M_1 and M_2 and of the five equilibrium points. The separation a between the two major bodies is the unit of distance and the five equilibrium points have the co-ordinates given In Table 1. This Table also gives the potential values W expressed in the dimensionless units of Eq. (5). We note that the lowest equilibrium point potential occurs at L_1 and increases progressively through L_2 and L_3, to become highest at L_4 and L_5. This progression can also be seen in Fig. 1 and 3, which both give a representation of the potential in the orbital plane.

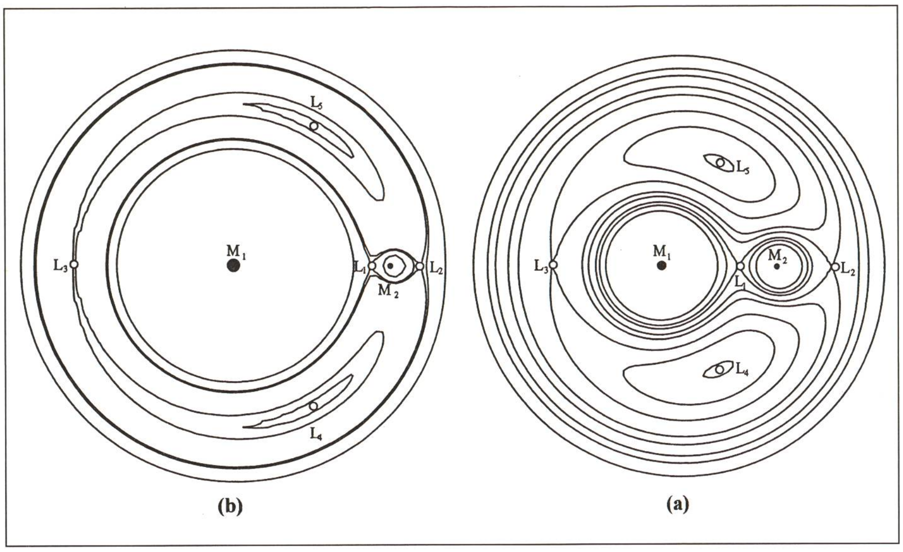

*Fig. 1. Contours of constant potential for the simplified three-body problem of Lagrange. There are five points, L_1 to L_5, where the tangential plane to the potential is horizontal. These points are called equilibrium points. Here they are numbered according to increasing potential values (cf. Table 1). The first three are saddle points whereas L_4 and L_5 are maxima. Under special conditions bodies of negligible mass can orbit L_4 and L_5 and these points then become libration centres. Graph (a) refers to the situation where the two heavy bodies have a mass ratio of 5, whereas (b) refers to a ratio of 100. The first three points are all located on the straight line passing through the two main objects and are often referred to as co-linear points. L_4 and L_5 are leading and trailing the lighter of the main bodies by 60°. With M_1 and M_2 they form equilateral triangles and are therefore often called equilateral points. In graph (a) the potential values at L_1 and L_2 are very similar and this leads to pairs of very close contours. Stable librations around L_4 and L_5 are possible only if the mass ratio of the two large bodies exceeds **24.96.** (see also Fig. 3)*

3. The co-linear equilibrium points L_1, L_2, and L_3

As said above, the location of the equilibrium points corresponds to the zeros of the first derivatives of the potential W. With the second derivatives we can then distinguish between saddle points and extrema. L_1 to L_3 turn out to be saddle points and are all located on the x-axis. Since this is an axis of symmetry the first derivative with respect to co-ordinate y vanishes everywhere on this axis. Any profile parallel to the y-axis is therefore extremal when it crosses the x-axis, and as seen in Fig. 1 and 3, all these extrema are in fact minima. The saddle points therefore exhibit maxima of the potential W when one moves along the co-linear x-axis, as seen in Fig. 4 for $q = 100$.

4. The equilateral libration points L_4 and L_5

As was already noted, L_4 and L_5 are locations where the potential is at a maximum, as shown in Fig. 4 and 5. At first it may seem surprising that it is precisely around the potential maxima that small bodies can librate. This is a consequence of the Coriolis force.

Let us imagine a small asteroid of mass m, initially located at L_4 or L_5. This mass will begin sliding down from the potential maximum. As it does so, the Coriolis force starts to act on the asteroid. This force is given by

$$\vec{F_c} = +2\,m\left(\vec{v}\times\vec{\omega}\right) \quad , \tag{6}$$

where $\vec{v}$ is the velocity with respect to the co-rotating co-ordinate system and $\vec{\omega}$ is the angular velocity of this rotating system with respect to an inertial reference frame. The Coriolis force increases with the distance and leads the asteroid into a complicated tadpole orbit [4] around the libration point. Cornish [5] suggests a picturesque analogy with the forces acting on air masses that rush into a low pressure system and begin to rotate because of the Coriolis forces, thus forming a stable vortex [6]. The rotation of the Earth clearly plays a major role in the tornado formation and the vortices always curl in the same direction in a given hemisphere, anti-clockwise when seen from above in the northern hemisphere, clockwise in the south. But the analogy is restricted to this action of the Coriolis force: the air masses that rush into the tornado begin at the ground with a large kinetic energy and then move upwards against gravity, exchanging kinetic for potential energy. The asteroid that slides off an L_4 or L_5 libration point starts its journey with a kinetic energy that increases from zero, being traded for potential energy

It is, of course, quite unlikely that an asteroid should, by chance, find itself at rest at an L_4 or L_5 libration centre to begin moving into a tadpole orbit. What will generally happen is that an asteroid may come shooting near a libration point. If it has an angular velocity of the appropriate size and direction, it will get trapped into a tadpole orbit. This explains why all the asteroids orbiting around L_4 and L_5 Lagrange libration points have tadpole orbits with angular velocities of the same sign. These angular velocities are all retrograde with respect to the orbit of the secondary body around the major one [4].

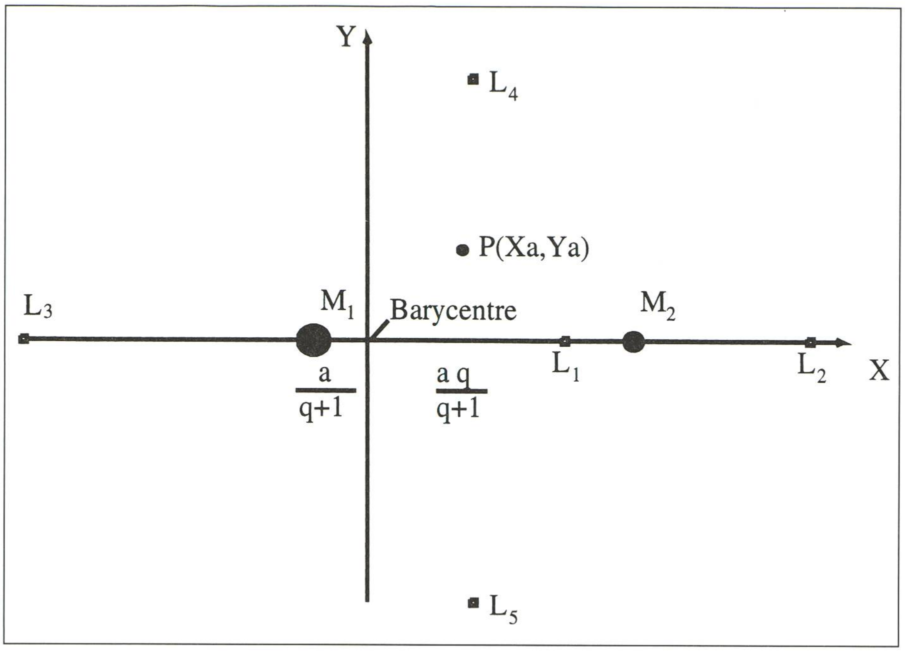

Fig. 2. Co-ordinate system chosen to derive the potential function. This system is fixed with respect to the barycentre, with which it is co-rotating. A mass ratio $q = M_1/M_2 = 5$ was chosen for this particular graph.

The potential can easily be extended to the three dimensions of space with an appropriate co-ordinate z, $V(x,y)$ becoming $V(x,y,z)$. It can then be shown that for motions perpendicular to the (x,y) - plane, an asteroid oscillates up and down with the same period as the secondary body's orbit, as one would in fact expect.

5. The «halo orbits» around the L_1, L_2, and L_3 co-linear points

It can also be shown that even with the Coriolis force the L_1 to L_3 co-linear points of Lagrange cannot become true *libration centres*. This is not always clearly stated. As an example, the interplanetary NASA probe **ISEE 3**, launched on 12 August 1978 was sent on what has been named a *halo orbit* around L_1, to observe the Sun; but it could stay in orbit around that point for four and a half years only with the help of on-board rocket motors. At the end of 1982 **ISEE 3** underwent a series of complicated manoeuvres and was diverted toward the exploration of several comets, among these comet Halley[1]. Quite recently, in August 2001, NASA sent the probe **GENESIS** on a similar *halo orbit* around L_1 where it should stay for some five years, collecting particles from the solar wind and return these to Earth for analysis.

6. Stability of the L_4 and L_5 libration points

We mentioned in Sec. 1 that orbits around the equilateral Lagrangian points are stable only if the mass ratio $q = M_1/M_2$ is greater than **24.96**. To demonstrate this, we shall have to consider the equation of motion of the asteroid in the co-rotating co-ordinate system.

The surface given by Eq. (1) or Eq. (5), and illustrated in Fig. 1 and 3, represents the potential function in which the asteroid moves with respect to a co-rotating co-ordinate system in which the line joining M_1 and M_2 is stationary. The potential surface includes the effect of the gravitational attraction of the two bodies M_1 and M_2, as well as the centrifugal force that always exists in a rotating reference frame. A third body of negligible mass m at another point will experience an acceleration that is equal to the negative of the gradient of the potential surface at that other point.

However, whenever a body moves, with velocity $\vec{v}$ relative to a rotating reference frame, it experiences, as well as the centrifugal force, the *Coriolis force*, which is described by Eq. (6) and is at right angles to the velocity $\vec{v}$. Thus, bearing in mind that the variables x and y are dimensionless quantities, and that real distances are xa, ya, etc. (cf. Sec. 1), the equations of motion in the x-y plane are:

$$\ddot{x} = -\frac{1}{a^2}\frac{\partial V}{\partial x} + 2\,\omega\,\dot{y} \quad , \qquad (7)$$

$$\ddot{y} = -\frac{1}{a^2}\frac{\partial V}{\partial y} - 2\,\omega\,\dot{x} \quad , \qquad (8)$$

We have to calculate the derivatives of V *in the vicinity of the Lagrangian equilateral points. At* these points the derivatives are of course zero, but near these points they can be approximated by Taylor's theorem

$$\frac{\partial V}{\partial x} = \left(x - x_0\right)\frac{\partial^2 V}{\partial x^2} \quad , \qquad (9)$$

$$\frac{\partial V}{\partial y} = \left(y - y_0\right)\frac{\partial^2 V}{\partial y^2} \quad , \qquad (10)$$

where the second derivatives are to be evaluated at the equilateral points and are not zero. Clearly some algebra is required in order to evaluate the derivatives from Eq. (1), but when this has been duly carried out the equations of motion in the vicinity of the Lagrangian points become

$$\ddot{\xi} - 2\,\omega\,\dot{\eta} = \omega^2\left[\tfrac{3}{4}\xi + \frac{\sqrt{27}\,(q-1)}{4(q+1)}\eta\right] \quad , \qquad (11)$$

$$\ddot{\eta} + 2\,\omega\,\dot{\xi} = \omega^2\left[\frac{\sqrt{27}\,(q-1)}{4\,(q+1)}\xi + \tfrac{9}{4}\eta\right] \quad . \qquad (12)$$

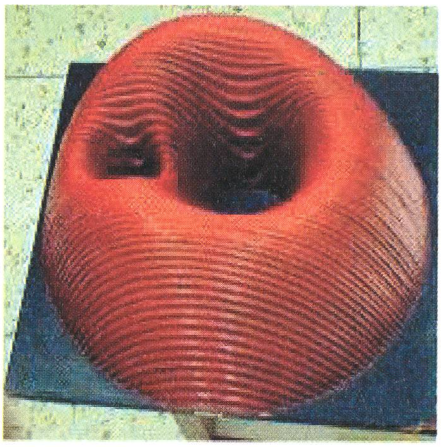

Fig. 3. Photograph of a model of the Lagrange potential function for a mass ratio $q = 5$. Note the infinitely deep troughs at the locations of the two major bodies and the decrease as the inverse squared distance far away from the barycentre. In this photograph the major mass M_1 is at the right of the secondary one, contrary to Fig. 1 and 2, where it is at the left. (Image courtesy of David D. Balam)

[1] More information on ISEE 3 is available on WEB site http://map.gsfc.nasa.gov/m_mm/ob_techorbit1.html

Here $\xi = x - x_0$ and $\eta = y - y_0$; in other words we have shifted the origin of coordinates from the barycentre to an equilateral Lagrangian point.

In order now to find the condition for stable orbits, we have to look for periodic solutions around the equilateral points; this means that we seek simple harmonic solutions of the form $\ddot{\xi} = -n^2 \cdot \xi$ and $\ddot{\eta} = -n^2 \cdot \eta$. The first time integrals of these are $\dot{\xi} = i\, n\, \xi$ and $\dot{\eta} = i\, n\, \eta$, where i is the imaginary unit. When these are substituted into Eq. (11) and (12), we obtain

$$\left(n^2+\frac{3}{4}\omega^2\right)\xi+\left[2\,\omega n i+\frac{\sqrt{27}}{4}\left(\frac{q-1}{q+1}\right)\omega^2\right]\eta=0 \quad , \qquad (13)$$

$$\left[2\,\omega n i-\frac{\sqrt{27}}{4}\left(\frac{q-1}{q+1}\right)\omega^2\right]\xi-\left(n^2+\frac{9}{4}\omega^2\right)\eta=0 \quad . \qquad (14)$$

Non-trivial solutions are possible only if the determinant of the coefficients of ξ and η is zero, which results in

$$n^4-\omega^2 n^2+\frac{27\, q\, \omega^4}{4\,(q+1)^2}=0 \quad . \qquad (15)$$

This is a quadratic equation in n^2, where n is the angular frequency of the orbit around the equilateral point. For real solutions, according to the theory of quadratic equations, $b^2 > 4\,a\,c$, which results in the inequality

$$q^2-25\,q+1>0 \quad , \qquad (16)$$

and the solutions to this are $q > 24.96$ or $q < 1/24.96$, which is what we set out to prove.

This is easily satisfied for the Sun-Jupiter system (q = 1047.5) and for the Earth-Moon system (q = 81). It has been suggested from time to time that emission lines seen in the spectrum of binary stars with mass ratios in the range of one to five might originate from gas accumulating at the Lagrangian points of these systems; but this cannot be so, for in that case orbits around the Lagrangian points are unstable.

If the potential had the circular symmetry implied by the developments leading to Eq. (15), we would be led toward two independent solutions, with a short and a long orbital period, both in the shape of circular orbits. However, the potential in the vicinity of the equilateral points has the shape of narrow curved ridges and the actual solutions therefore take the much more complex form of tadpole orbits [4] mentioned before.

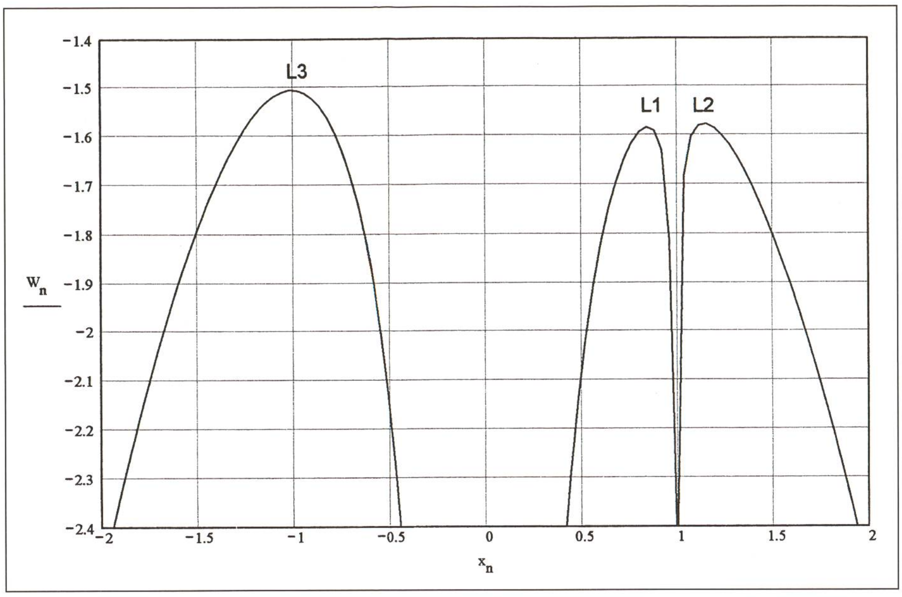

Fig. 4. Plot of the potential function W along the x-axis for a mass ratio $q = M_1/M_2 = 100$. Because of the fairly high mass ratio M_1 is located close to x = 0 and M_2 close to x = 1. At these positions the potential tends to - ∞. Note that $W(L_1) < W(L_2) < W(L_3)$. (cf. Table 1)

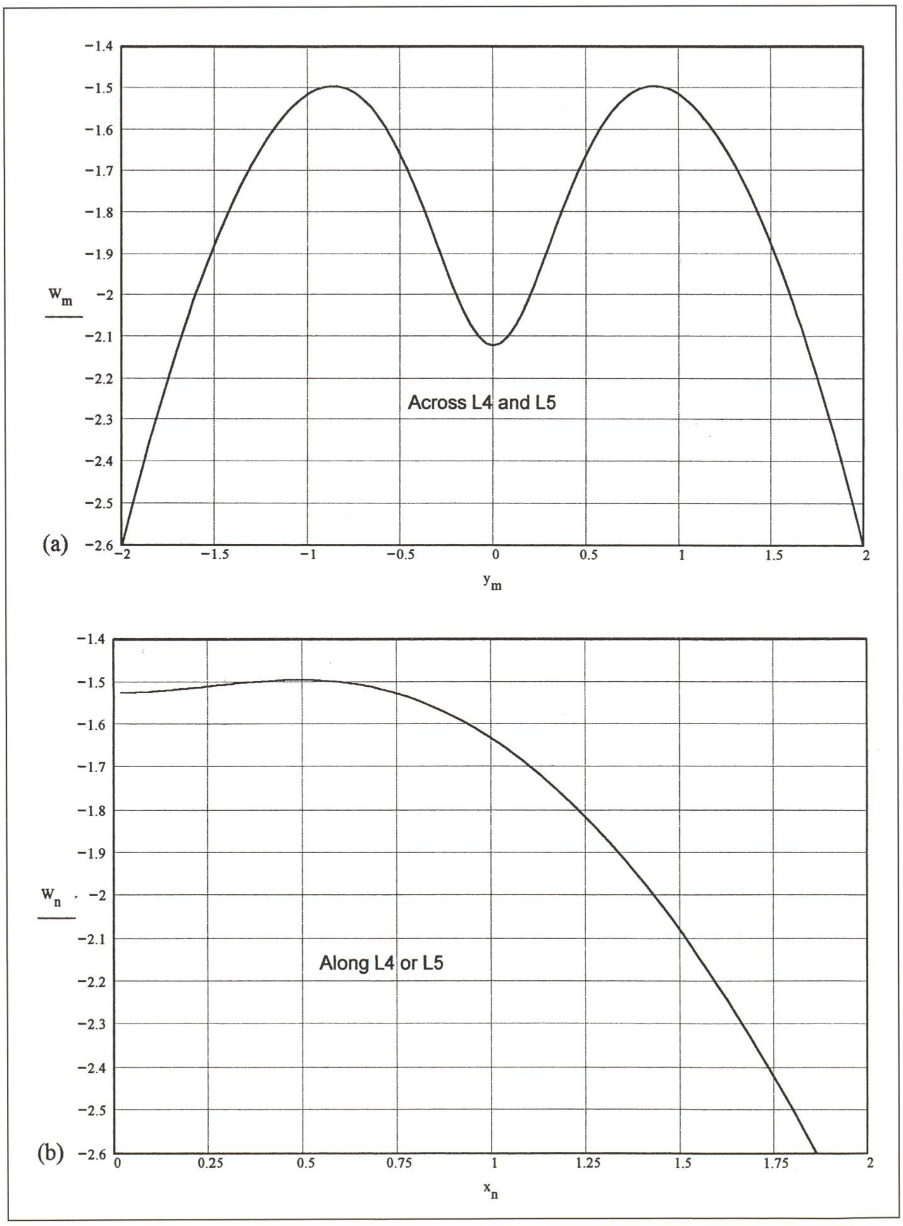

Fig. 5. (a) Plot of the potential function W across the line from L_4 to L_5 for a mass ratio q=100. (b) Similar plot along lines through L_4 or L_5 parallel to the x – axis. Note that the potential is the highest of all at L_4 and L_5. (cf. Table 1)

	x	y	q = 5	q = 24.96	q = 100
L_1	$x=1-\left[\frac{1}{3\cdot(q+1)}\right]^{1/3}$	$y=0$	$x=0.491889$ $(x=0.618429)$ $y=0$ $W=-1.874495$	$x=0.744935$ $(x=0.765834)$ $y=0$ $W=-1.682581$	$x=0.848624$ $(x=0.851115)$ $y=0$ $W=-1.583321$
L_2	$x=1+\left[\frac{1}{3\cdot(q+1)}\right]^{1/3}$	$y=0$	$x=1.271410$ $(x=1.381571)$ $y=0$ $W=-1.768170$	$x=1.214439$ $(x=1.234167)$ $y=0$ $W=-1.657078$	$x=1.146320$ $(x=1.148886)$ $y=0$ $W=-1.576726$
L_3	$x=-1-\frac{5}{12\cdot(q+1)}$	$y=0$	$x=-1.069165$ $(x=-1.069144)$ $y=0$ $W=-1.582524$	$x=-1.016047$ $(x=-1.016050)$ $y=0$ $W=-1.519239$	$x=1.004125$ $(x=-1.004125)$ $y=0$ $W=-1.504949$
L_4	$x=\frac{1}{2}\cdot\frac{(q-1)}{(q+1)}$	$y=+\frac{\sqrt{3}}{2}$	$x=0.333333$ $y=0.866025$ $W=-1.430556$	$x=0.461479$ $y=0.866025$ $W=-1.481482$	$x=0.490099$ $y=0.866025$ $W=-1.495099$
L_5	$x=\frac{1}{2}\cdot\frac{(q-1)}{(q+1)}$	$y=-\frac{\sqrt{3}}{2}$	$x=0.333333$ $y=-0.866025$ $W=-1.430556$	$x=0.461479$ $y=-0.866025$ $W=-1.481482$	$x=0.490099$ $y=-0.866025$ $W=-1.495099$

Table 1: Co-ordinates of the Lagrange equilibrium points and potential values expressed in terms of W as defined in the text, for mass ratios q of 5, 24.96 and 100 for the two main bodies. The formulae for co-ordinate x of the co-linear points L_1, L_2 and L_3 are only first order approximations, with the corresponding figures given in brackets. The unbracketted figures, on the other hand, are the true co-ordinates of these points.

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