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Beyond Pythagoras: Ancient Techniques for Designing Musical Instrument Scales

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Over the last decade or so, a new trend has emerged in the reconstruction of historical musical instruments: "copying" based on "first principles". That is, the modern maker doesn't slavishly reproduce each and every detail of an original instrument, but rather tries to understand the design and construction principles which guided the original maker. By reproducing the *process*, we produce objects which – though new – remain within the confines of a historical tradition, exactly as the modern "historically-informed" musician does when performing a piece of "early music".

The challenge, of course, is figuring out exactly what these principles were. Any object can be described by a number of design theories. Which theory we consider most plausible depends greatly upon our education and experience, to say nothing of our broader view of existence – how we think about and manipulate the materials and forces which confront us in the "real" world. As late-20th century people, we cannot assume that our modes of thought are similar to those of the skilled craftsman of two or three hundred years ago. Therefore, it is of fundamental importance to learn as much as we can about their techniques, both intellectual and practical. Only by understanding and approaching the problems of instrument making in a similar manner will we gain real insight into their creative processes.

To this end, a number of articles have recently appeared on the geometrical layout techniques used by old makers to design the cases of harpsichords and fortepianos. However, one critical aspect of instrument design has yet to be subjected to such an approach: the design of an instrument's scale. Basic Pythagorean proportions[1] are still assumed to be the fundamental principle underlying the determination of string lengths for most instruments. However, a surprising number cannot be explained with Just scaling, a fact which remains largely unacknowledged, and therefore unexplained by modern organology. This is partly due to the use of methods of examining scales

1 Regarding scaling, "Pythagorean proportions" means a scale in which the octaves halve or double precisely in length, and the length of each successive note is related to the previous by the 12th root of 2. Such a scale is also called "Just".

which do not make significant aberrations apparent.[2] Secondly, when deviations from Just scaling *are* noticed, organologists seem to be incapable of imagining methods of devising such scales other than minor variations upon the simplistic theme of Pythagorean proportions. Dogma has also gotten in the way, in a form which I call "The Tyranny of the Whole Zoll".[3] This is my term for an approach based on an assumption of Pythagorean scaling in combination with "rediscovering" the local/temporal measuring unit used by the ancient maker. Such an approach inevitably results in forcing the observed string lengths into integer units or simple fractions of the assumed original measure. While this works for some instruments, it fails miserably at explaining a significant number of others.

Modern organology has settled into the rather complacent view that the scales of stringed keyboard instruments can be divided into two broad categories:

(1) Pythagorean scaling throughout (or nearly so); most often found in instruments of southern building schools, and generally assumed to be strung entirely in brass (see Chart 1).

(2) Pythagorean scaling in the treble half of the instrument only, often from around middle c up, but with a "foreshortened" tenor and bass; thought to be typical of "northern" building schools, and usually (though not always) assumed to be strung in brass in the bass and iron from the high tenor up (see Chart 2).

In either instance, it is no mystery how a builder would have designed the Pythagorean portions of the scale; beginning with a design length – usually assumed to be c2 – one simply doubles and halves to get the octaves. If a second reference note is desired, either f or g can be determined by simple Pythagorean proportions.[4] These lengths were marked directly on the soundboard, starting from the nut, and then used to position the bridge during gluing. The bridge itself was either sawn or bent into a curve which best connected the marks of

2 Log10 graphing of string lengths is far too coarse to make aberrations of 1 or 2 semitones visually obvious, especially in the treble, where such deviations are often found.

3 *Zoll* is the German word for "inch", also called *duim* in Dutch and *pouce* in French. Regardless of regional variations in name and size, the "inch" of approximately 24 to 27 mm was the standard unit of measure throughout Europe before the introduction of the metric system.

4 Any f will be half again as long as the c above it, or any g will be a third less the length of any c below it. These are, of course, pure Pythagorean "harmonic" proportions, which give lengths slightly different from a true "Pythagorean scale" in which the proportion of every step is the 12th root of 2. However, for the practical purposes of designing a scale and positioning a bridge upon a soundboard, this error is insignificant. In any event, as I'll later demonstrate, an extremely simple method also exists for deriving the length of a true "Pythagorean" f# (related to c lengths by the square root of 2).

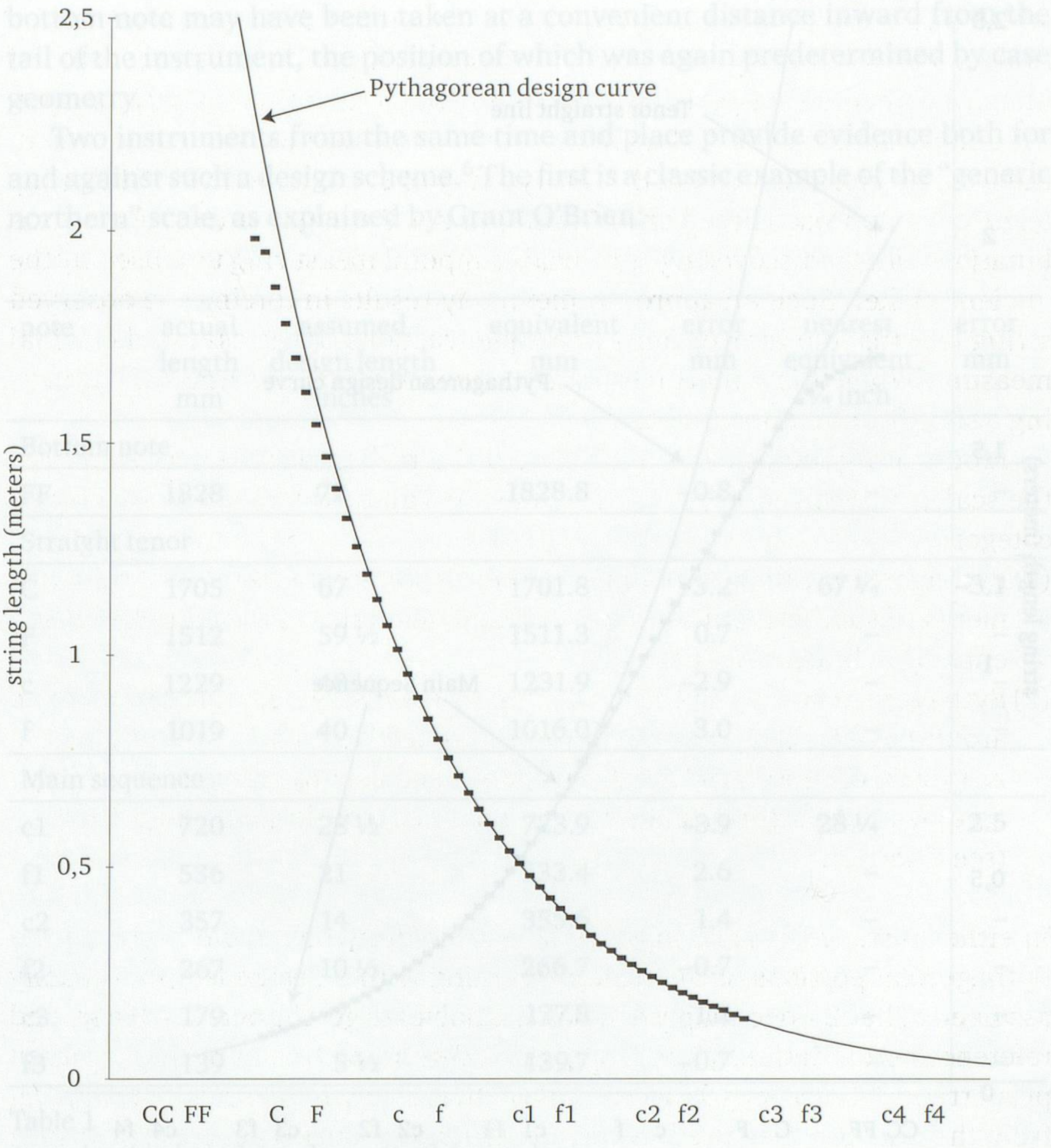

Chart 1 – "Typical southern" scale shape. String lengths of the Cristofori 1722 harpsichord (Leipzig # 84).

the "reference" notes. In a "generic northern" instrument, such logic would have been used by the builder to design only the treble portion of his scale, down to the point at which foreshortening begins. I call this area the "Main Sequence", since it is here that we find the basic length/stress/pitch identity of the instrument.[5]

5 Scale length determines the stress upon the wire (closeness to breaking) at any given pitch level. Builders are of course free to choose a scale much too short for the desired pitch, which would produce a stress level well below breaking. The only thing we can determine with any degree of certainty is the maximum pitch level an instrument with a certain scale length can withstand.

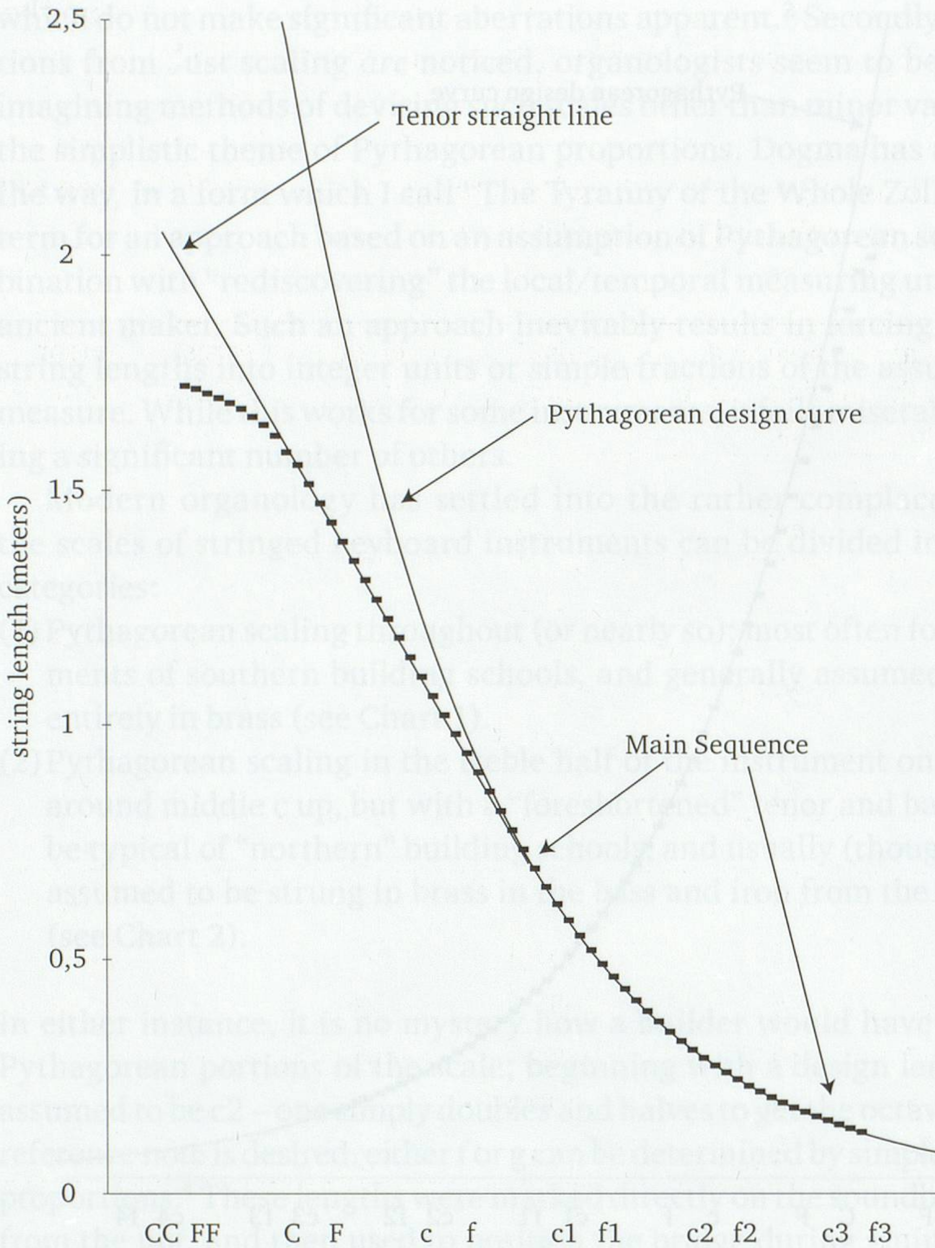

Chart 2 – "Typical northern" scale shape. String lengths of the Kirckman 1771 harpsichord (Deutsches Museum #87–409).

Below the Main Sequence, the tenor bridge often followed a straight (or near-straight) line, quite often parallel to the bentside. The angle of the bentside itself was probably predetermined by case layout geometry, meaning that the string lengths in the foreshortened area were determined by extra-acoustical considerations. Some builders let the bridge continue along this line to the very bottom; others chose an even more severely foreshortened bottom note length by bending the bridge away from the straight line, eventually arriving at a predetermined bottom note length. As with the tenor straight line, this bottom note may have been taken at a convenient distance inward from the tail of the instrument, the position of which was again predetermined by case geometry.

Two instruments from the same time and place provide evidence both for and against such a design scheme.[6] The first is a classic example of the "generic northern" scale, as explained by Grant O'Brien:

note	actual length mm	assumed design length inches	equivalent mm	error mm	nearest equivalent ¼ inch	error mm
Bottom note						
FF	1828	72	1828.8	–0.8	–	–
Straight tenor						
C	1705	67	1701.8	3.2	67 ¼	-3.1
F	1512	59 ½	1511.3	0.7	–	–
c	1229	48 ½	1231.9	–2.9	–	–
f	1019	40	1016.0	3.0	–	–
Main sequence						
c1	720	28 ½	723.9	–3.9	28 ¼	2.5
f1	536	21	533.4	2.6	–	–
c2	357	14	355.6	1.4	–	–
f2	267	10 ½	266.7	–0.7	–	–
c3	179	7	177.8	1.2	–	–
f3	139	5 ½	139.7	–0.7	–	–

Table 1
Actual string lengths and proposed design logic
Double manual harpsichord by Johannes Willbrock, London, 1730

The left three columns are taken from the published article, the right represent my analysis of the accuracy of the interpretation. The proposed design lengths for this instrument are generally believable, though not without some reservations. O'Brien offers some lengths which are not the best "¼ inch fit", for reasons which remain unexplained. Bass C, for example, is almost exactly half-way between 67" and 67 ¼" and could arguably go either way. Neither

6 Grant O'Brien, *The Double-Manual Harpsichord by Francis Coston, London, c. 1725*, The Galpin Society Journal XLVII (March 1994), pp. 2–31.

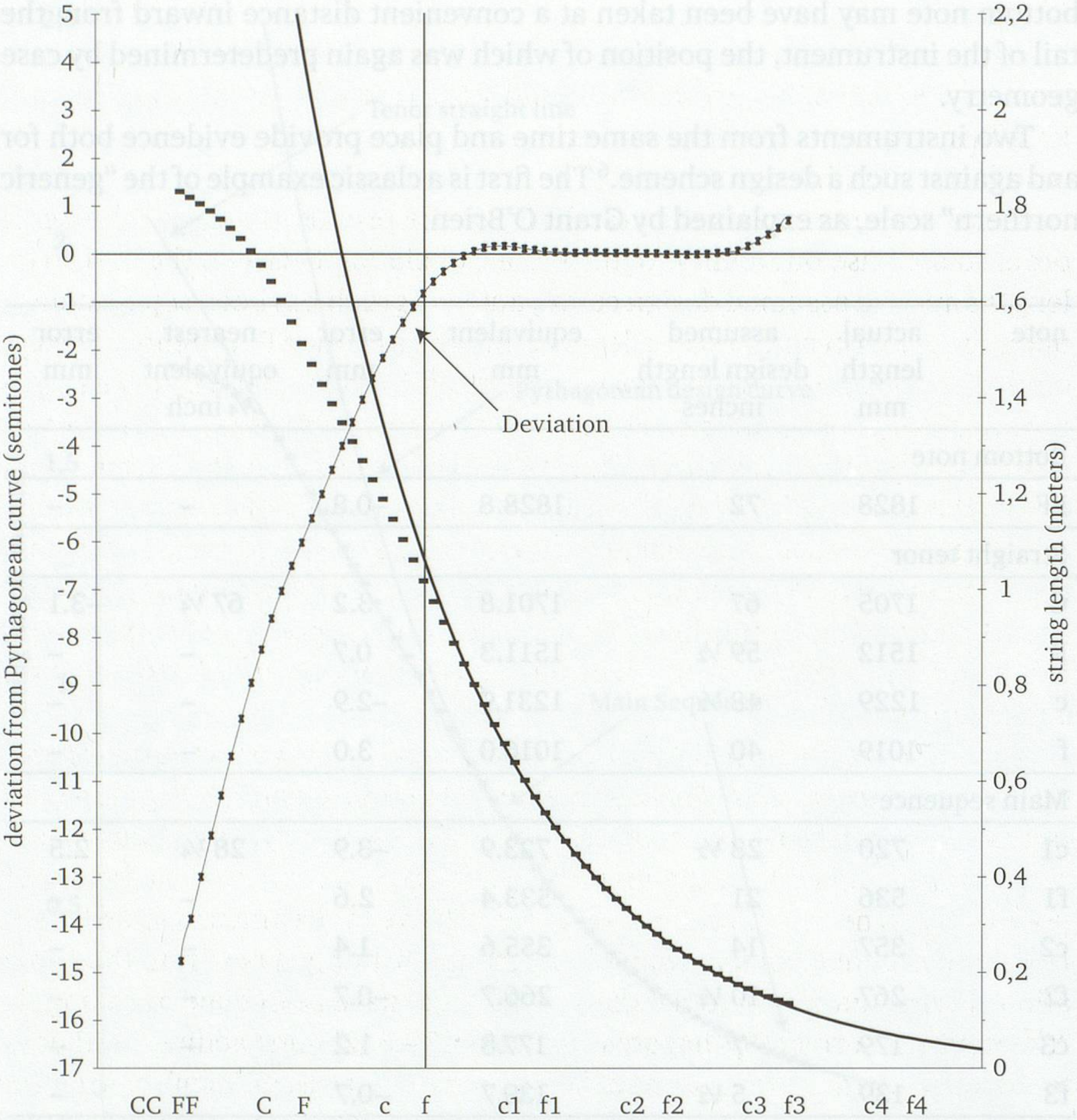

Chart 3 – Real string lengths and proportional deviation of the Willbrock harpsichord.

inch equivalent fits any sort of apparent mathematical design scheme, though as stated above, I don't believe builders actually "designed" the lengths of any of the notes in the straight tenor section in any acoustical or proportional (octave) sense.

In the treble, O'Brien suggests 28 ½" for middle c, when 28 ¼" would have been a closer fit. There is no apparent advantage to 28 ½" – in fact, we can probably assume that the builder's original intent was 28" (711.2 mm), because it fits a scheme of pure Pythagorean proportions for all f's and c's from c1 to c3. While the *length* error required to accept this interpretation may seem large – almost 9 mm – we shouldn't forget that the bridge at c1 is running at an oblique angle to the strings. Therefore, the amount of actual perpendicular *bridge displacement* (during gluing, or due to age and distortion) needed to cause this length error is quite small: only about 2.5 mm. In this register, an error of this magnitude between the "ideal" length and the actual length is readily acceptable, since the length aberration produces an *acoustic error* (i.e. a change in tension/stress) which is negligible.

Chart 3 demonstrates the credibility of the proposed design scheme. In order to increase our ability to judge the correlation between the real string lengths and the assumed design curve, another graph has been superimposed upon the graph of actual string lengths and the assumed Pythagorean design curve. This second graph shows the deviation from the design curve in proportional units of semitones. For example, at tenor f we see that the length is almost 1 semitone too short. This means that tenor f has the length which tenor f♯ would have had, had the scale not been foreshortened and continued following the Just curve. Such an analysis makes it easy to see deviations in the high treble which are small in an absolute sense but nonetheless important acoustically, while reminding us that exactly the opposite holds true in the tenor and bass, that is, fairly large absolute deviations are of little acoustic significance.[7]

For the most part the scale does indeed follow the Pythagorean curve mapped out by the design lengths. The slight "error" at middle c causes a slight increase in scale length relative to the ideal, though being about 1/10th semitone, it remains insignificant. In the high treble, the scale seems to have been purposely lengthened by almost one semitone relative to the ambient scale length. Perhaps the builder simply wanted to keep the 8' bridge back from the gap to allow more room for the top of the 4'. This could have been done by taking the "theoretically correct" Pythagorean value for f3 – 5 ¼" – and adding another ¼" to the length. In any event, we have little trouble accepting O'Brien's conclusion: "Clearly Willbrock was … using the inch and the inch divided into halves and quarters to design the scalings of his harpsichord." Beginning with a c2 of 14 inches, Willbrock could have easily calculated the other reference lengths with very simple arithmetic.

Unfortunately, O'Brien's analysis of the other instrument is much less convincing:

7 The amount by which the length of any string must be altered in order to introduce an acoustic effect of a certain magnitude (changes in stress, inharmonicity, etc.) is always proportional to the string's length, and cannot be expressed in an absolute sense.

note	actual length mm	proposed design length inches	equivalent mm	error mm	nearest equivalent ¼ inch	error mm
Bottom note						
GG	1689	66 ½	1689.1	–0.1	–	–
Straight tenor						
C	1552	61	1549.4	2.6	–	–
F	1346	53	1346.2	–0.2	–	–
c	1063	42	1066.8	–3.8	41 ¾	2.5
f	859	34	863.6	–4.6	33 ¾	1.8
Main sequence						
c1	603	24	609.6	–6.6	23 ¼	–0.3
f1	460	18	457.2	2.8	–	–
c2	314	12 ½	317.5	–3.5	12 ¼	2.9
f2	239	9 ½	241.3	–2.3	–	–
c3	163	6 ½	165.1	–2.1	–	–
e3	139	5 ½	139.7	–0.7	–	–

Table 2
Actual string lengths and assumed design logic
Double manual harpsichord by Francis Coston, London, c. 1725

Note that there are two curious choices of proposed design lengths in the treble; why should we reject the nearest ¼ inch equivalents for c1 and c2, when in both cases they represent smaller deviation from the actual string lengths? Accepting a larger assumption of error might well be valid if the resultant inch values more closely followed Pythagorean proportions (or some other readily apparent design scheme), but they don't. Perhaps O'Brien assumed that Coston was using some kind of *modified* Pythagorean scheme, with a basic design series of 24, 18, 12, 9, 6, but altered from c2 upward using a "constant addition" method: Just values plus ½". However, this conflicts with O'Brien's stated conclusion that the scale is "based on a c2 of 12 ½ inches."

If we accept this c2 length as the genesis of the whole series, how would the lengths above and below have been generated? How did Coston determine his length for the top note e3? No explanation whatsoever is offered for the *design method* of this scale, though O'Brien does admit that the "8' treble scalings are not strictly Pythagorean." Despite this, he nonetheless confidently concludes:

> From these measurements it can be seen that, as one might have expected, Coston was using the inch as his unit of length, and that the c and f strings were designed to be either whole numbers … or integral plus half- and quarter-divisions of inches.

This is a classic example of the "Tyranny of the Whole Zoll" at work: the observed data is freely interpreted in a manner which fits the *expectation* of rational numbers, and the result is offered as some sort of explanation of design principles. We shouldn't forget that *any* metric value will always be within ±3.2 mm of some ¼ inch value, and simply converting millimeter measurements to their nearest (or not) simple-fraction inch equivalents proves absolutely nothing – *not unless the result gives us a believable explanation for how the builder actually went about the work of devising the scale.*

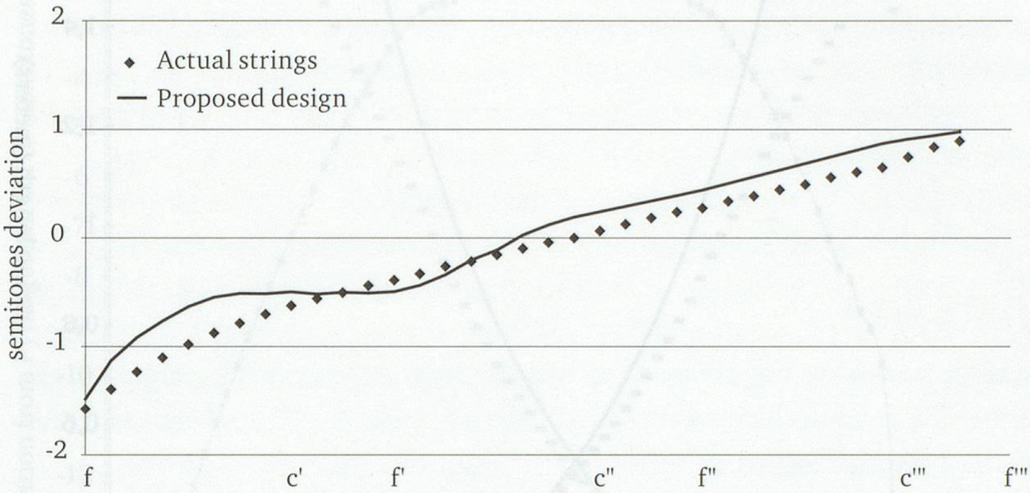

Chart 4 – Proposed design curve compared with actual string lengths of the Coston harpsichord. Zero line represents a Pythagorean curve at the actual c'' length.

A graphic analysis illustrates the unlikelihood that the proposed design lengths represent the original method by which Coston created this scale. Chart 4 shows the deviation from a Pythagorean curve of two spline curves (Main Sequence only): the curve connecting the real string lengths at all f's and c's, and the curve which connects O'Brien's proposed design notes.[8] Leaving aside the question of how Coston might have arrived at these values, note that the "reference lengths" do not provide believable coordinates for controlling the bridge shape. On the contrary, it appears that "design" and "practice" have exchanged places; usually the design notes indicate a smoothly-flowing curve which the actual bridge follows only approximately, as we saw above with the Willbrock. But in this case, the proposed *ideals* describe a series of points which produce a curve which has a noticeable bump at f1. The actual string lengths, however, indicate a distinct "vector" of scaling logic with an amazing degree of precision, as shown by the near-perfect straightness of the upwardly-sloping deviation trace, from middle c all the way up to the top note.[9] What is the logic

8 A "spline" curve is a curve which follows the path which a real flexible object – such as a bent stick – would follow when it is bent so as to intersect a series of points.

9 The instrument has no d#3 key, so in the graph, the note e3 is positioned where d#3 should be, just as it is in the instrument itself.

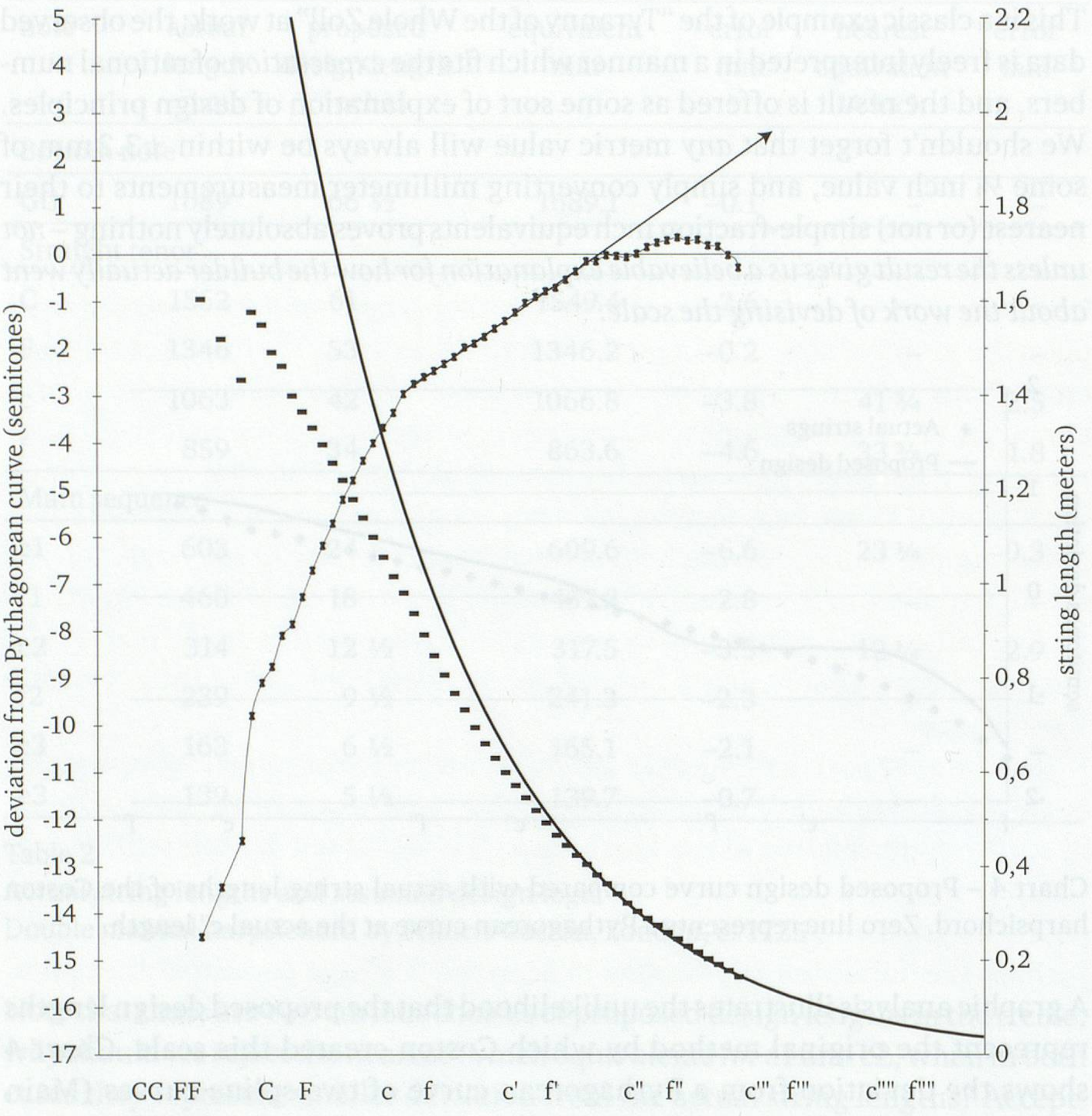

Chart 5 – Real string lengths and proportional deviation of the Tibaut 1691 harpsichord.

behind this "scale vector"? How could an 18th century builder have designed such a scale, and what methods would he have used to mark the string lengths on his soundboard for positioning the bridge? Why would he have wanted such a non-Pythagorean scale in the first place? The answers to these questions are precisely what this workshop is all about.[10]

10 While the use of reference lengths/marks is already well-documented, the reader will note that I have made little effort to discover any evidence that my proposed scale calculation methodology was actually used by string keyboard instrument makers (with one exception – see below). My purpose is merely to provide a believable hypothesis, using known tools and techniques of the times, which offer a credible explanation for a number of enigmatic scales.

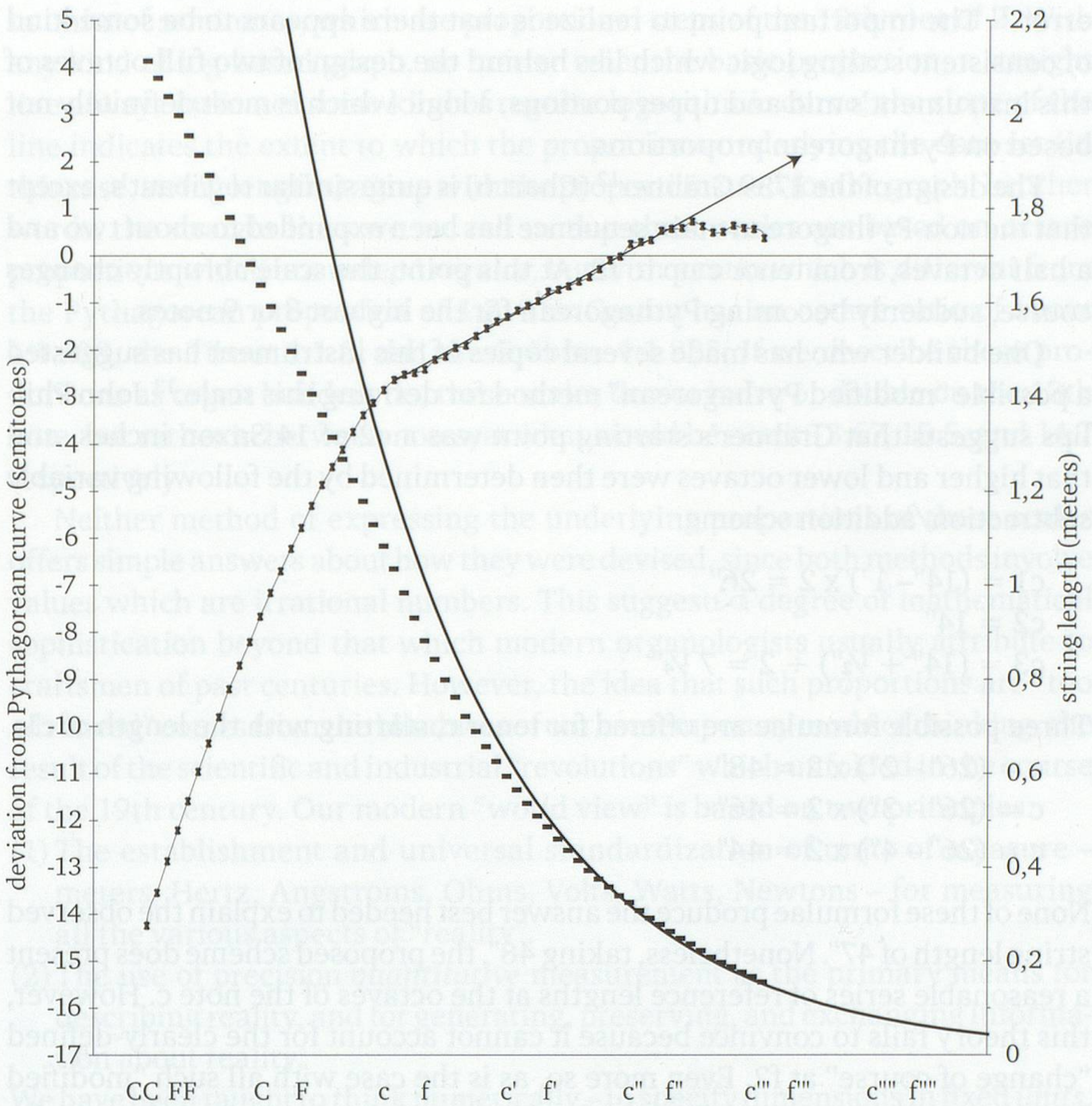

Chart 6 – Real string lengths and proportional deviation of the Gräbner 1739 harpsichord.

Before we proceed, I think it is important to demonstrate that this kind of scale is by no means unique to one instrument. Charts 5 and 6 illustrate two more northern harpsichords with similar non-Pythagorean scales.

A large portion of the main sequence of the Tibaut in the Paris Conservatory (Chart 5) also exhibits a deviation trace which follows a distinct scaling vector, indicated here by the upward-sloping arrow, as opposed to the horizontal trace of a Pythagorean curve. Note that this slope is much steeper than in the Coston instrument. This area runs from about tenor d up to about c2, where the logic appears to level off and follow Pythagorean proportions. The slight "bubble" at f2 may be intentional, but it could also be due to a combination of age, original workshop error, and/or modern measurement error.[11] The important point to realize is that there appears to be some kind of consistent scaling logic which lies behind the design of two full octaves of this instrument's mid and upper portions, a logic which is most definitely not based on Pythagorean proportions.

The design of the 1739 Gräbner[12] (Chart 6) is quite similar to Tibaut's, except that the non-Pythagorean main sequence has been expanded to about two and a half octaves, from tenor c up to f2. At this point, the scale abruptly changes course, suddenly becoming Pythagorean for the highest 8 or 9 notes.

One builder who has made several copies of this instrument has suggested a possible "modified Pythagorean" method for deriving this scale.[13] John Phillips suggests that Gräbner's starting point was a c2 of 14 Saxon inches, and that higher and lower octaves were then determined by the following variable subtraction/addition scheme:

c1 = (14"– 1") x 2 = 26"
c2 = 14"
c3 = (14" + ½") ÷ 2 = 7 ¼"

Three possible formulae are offered for tenor c, starting with the length of c1:
c = (26" – 2") x 2 = 48"
c = (26" – 3") x 2 = 46"
c = (26" – 4") x 2 = 44"

None of these formulae produce the answer best needed to explain the observed string length of 47". Nonetheless, taking 48", the proposed scheme does present a reasonable series of reference lengths at the octaves of the note c. However, this theory fails to convince because it cannot account for the clearly-defined "change of course" at f2. Even more so, as is the case with all such "modified Pythagorean" mathematical methods, this proposal only produces lengths for octaves of the starting note. Should the builder want other reference lengths as well – octaves of f, g, or f♯ (Ruckers), for example – to help position the bridge more accurately or to define a top note, such schemes provide no means whatsoever for calculating them.

The first step to unraveling this enigmatic situation is to realize the significance of the straight-line scaling vectors seen on these instruments. The deviation graphs used here are a specialized form of logarithmic graph, plotted

11 Rounding to the nearest millimeter, a common practice in measurement taking, can cause the appearance of aberrations in the highest treble registers, regardless of whether or not they actually exist.

12 Kunstgewerbemuseum, Schloß Pillnitz, Dresden, Inv. No. 37414.

13 See John Phillips, *The Surviving Harpsichords of the Gräbner Family*, forthcoming in the proceedings of the Stiftung Kloster Michaelstein. John very kindly provided me with a prepublication version of this article, as well as with some additional string lengths not contained therein.

in units of semitones, which are proportional steps of the 12th root of 2. With any kind of log-based graph, no matter what the basic proportion, a straight line always indicates some kind of regular logarithmic curve; the slope of the line indicates the extent to which the proportions underlying the data set (in this case, string lengths) agree with the proportion used for the graph. In other words, the straight lines we see tell us that these scales are based on octave proportions which are indeed *regular*, but with a ratio which is different from the Pythagorean proportion of 1:2. The Coston has an octave ratio of about 1:1.928, the Tibaut 1:1.8, and the Gräbner 1:1.835. If we describe these proportions as organ builders do, these scales do not halve or double on the 13th note, but rather in between notes, at what would be notes 13.67, 15.5, and 14.7 respectively.

Neither method of expressing the underlying proportions of these scales offers simple answers about how they were devised, since both methods involve values which are irrational numbers. This suggests a degree of mathematical sophistication beyond that which modern organologists usually attribute to craftsmen of past centuries. However, the idea that such proportions are "too advanced" comes from the blindness of our contemporary mode of thinking, the result of the scientific and industrial "revolutions" which unfolded in the course of the 19th century. Our modern "world view" is based on two principles:

(1) The establishment and universal standardization of units of measure – meters, Hertz, Angstroms, Ohms, Volts, Watts, Newtons – for measuring all the various aspects of "reality".

(2) The use of precision *quantitative* measurement as the primary means for describing reality, and for generating, preserving, and exchanging information about reality.

We have been taught to think numerically – to specify dimensions in fixed *units*, and to manipulate these units with mathematical calculation. Our fixation upon "number crunching" blinds us to the methods used by ancient tradesmen, whether architects, ship builders, ordinance designers, or musical instrument makers. For them, universally-consistent absolutes did not exist, nor did the "thinking machines" which make complex numerical calculation fast and effortless. Their world consisted of *proportions* between things. Their methods of calculation were based on geometry, and their "calculators" were the straight edge, the square, and the compass.[14]

14 For example, Thomas Bradwardine stated (c.1320) that a geometrical approach was the best manner of dealing with proportions because it solves the problem of irrational numbers, "… for every proportion has to do with dimensions, but not every proportion is expressible with numbers [… *quia omnis proportio est magnitudinis sed non omnis est numeralis*]." See A. G. Holland, *An Examination of Bradwardine's Geometry*, Archive for History of Exact Sciences 19, 1978, p. 151.

The organ is the most complex of all acoustic musical instruments ever devised by man, a status which it had already achieved many hundreds – if not thousands – of years ago. The construction of even a relatively small instrument involves the production of hundreds of precisely-proportioned sound-producing objects. By comparison, designing the scale of a stringed keyboard instrument is child's play. Not surprisingly, organ building traditions can provide us with both the theory and techniques for understanding and manipulating scales which appear to be based upon sophisticated mathematics.

Organ builders discovered long ago that pipes could not be scaled according to Pythagorean proportions; if several octaves are made with diameters that double or halve on the 13th pipe, the sounds produced by the whole rank are out of balance, the bass being too thick and heavy, the treble shrill and weak.[15] Furthermore, the extreme registers could exhibit speech problems, such as poor tone, unstable attack, garbled or wobbly sustain, and a tendency to overblow. These problems are eliminated when non-Pythagorean octave proportions are used; the scale is tapered so that pipe diameters increase (going down into the bass) and decrease (going up into the treble) *less rapidly* than with "theoretically correct" proportions. In other words, bass pipes are too thin and treble pipes are too fat. This is directly analogous to a harpsichord or piano scale in which the bass strings are too short and the treble strings are too long relative to their theoretical Pythagorean lengths based on some note near the middle of the compass. For centuries, organ builders have used two types of geometrical templates both to devise such scales and to mark out the objects under construction in the workshop. These graphical calculating tools are the right triangle and the spline curve.

The first type of template is constructed by drawing a right triangle in which the desired *step* (not octave) proportion exists between the base and hypotenuse. To illustrate the principle, here is the recipe for constructing a Pythagorean template:

15 Organ builders, unlike stringed keyboard makers, do not "scale" (i.e. consciously design) the lengths of pipes. Their primary design concern is cross-sectional area, followed by other matters affecting tone quality, such as mouth width, cut-up, foot hole diameter, etc., all of which follow different scalings for each type of pipe (flute, diapason, string, etc.) in each division (*Hauptwerk*, *Rückpositiv*, *Oberwerk*, etc.). Length is simply allowed to become whatever it becomes depending upon wind pressure and tonal scaling; it is precisely determined only during the final assembly and voicing of the instrument. This is why pipes of different ranks which speak the same note in the same register (all 8' pipes for the note C, for example) are *not* all of the same length. Ancient pipe makers probably judged the length of the raw pipe by experience, at least until the mid-19th century, when the French organ builder Cavaillé-Coll devised a formula for predicting length (more or less) based upon wind pressure and area.

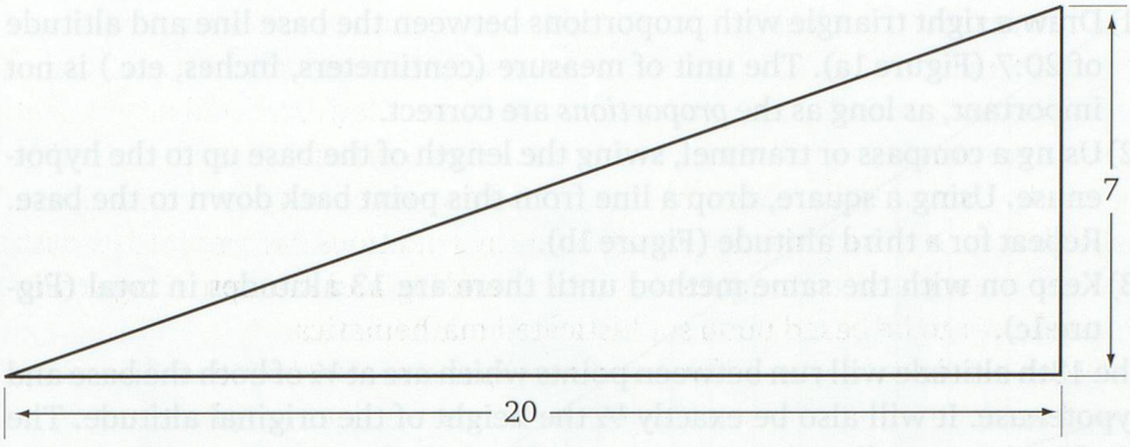

Figure 1a – Constructing the basic triangle.

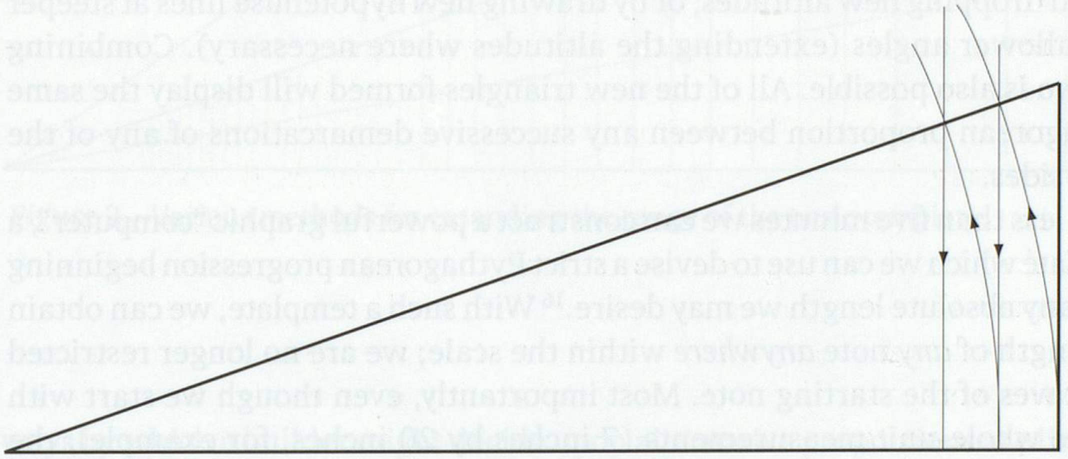

Figure 1b – Dividing the base line into proportional units.

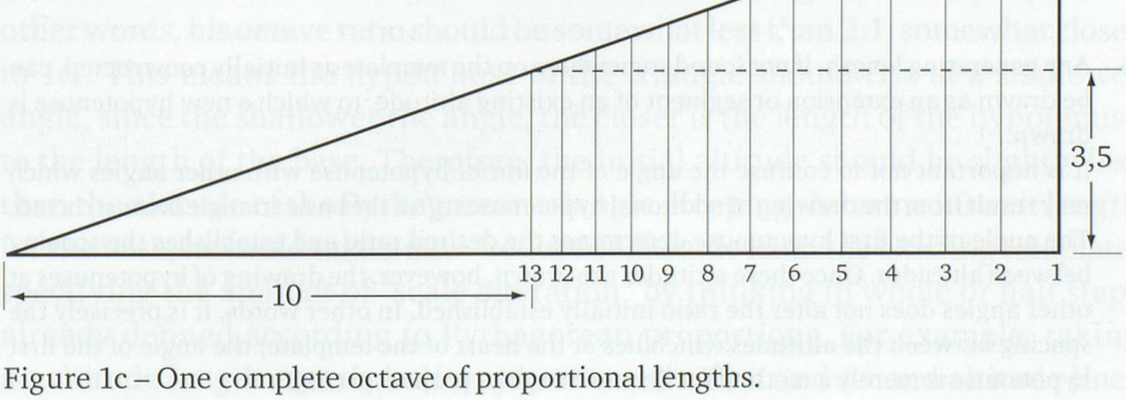

Figure 1c – One complete octave of proportional lengths.

(1) Draw a right triangle with proportions between the base line and altitude of 20:7 (Figure 1a). The unit of measure (centimeters, inches, etc.) is not important, as long as the *proportions* are correct.
(2) Using a compass or trammel, swing the length of the base up to the hypotenuse. Using a square, drop a line from this point back down to the base. Repeat for a third altitude (Figure 1b).
(3) Keep on with the same method until there are 13 altitudes in total (Figure 1c).

The 13th altitude will run between points which are at ½ of both the base and hypotenuse. It will also be exactly ½ the height of the original altitude. The 13 altitudes, as well as the 13 segments of both the hypotenuse and base they define (taken from the left point of the triangle), will reduce by a consistent ratio of the 12th root of 2. Once the basic template is constructed, its range can be increased by either continuing the same system of swinging the base lengths up and dropping new altitudes, or by drawing new hypotenuse lines at steeper or shallower angles (extending the altitudes where necessary). Combining the two is also possible. All of the new triangles formed will display the same Pythagorean proportion between any successive demarcations of any of the three sides.

In less than five minutes we can construct a powerful graphic "computer", a template which we can use to devise a strict Pythagorean progression beginning with any absolute length we may desire.[16] With such a template, we can obtain the length of *any* note *anywhere* within the scale; we are no longer restricted to octaves of the starting note. Most importantly, even though we start with simple whole-unit measurements (7 inches by 20 inches, for example), the majority of the derived lengths will be irrational numbers – a fact which is of no consequence whatsoever, *precisely because we have no need to use a standard measuring stick*. We can transfer the lengths of the desired reference notes from the template to the instrument under construction by copying them onto a "scaling stick", a "dedicated ruler" so to speak. Using such a template, we are completely free of the Tyranny of the Whole Zoll.

Making a triangle for *non*-Pythagorean proportions is just as easy; we need only begin with a different ratio between base and altitude.[17] The number of

16 Any generating length, if not found somewhere on the template as initially constructed, can be drawn as an extension or segment of an existing altitude, to which a new hypotenuse is drawn.

17 It is important not to confuse the angle of the *initial* hypotenuse with other angles which may result from the drawing of additonal hypotenuses *after* the basic triangle is constructed. The angle of the first hypotenuse determines the desired ratio and establishes the spacing between altitudes. Once these altitudes are drawn, however, the drawing of hypotenuses at other angles does not alter the ratio initially established. In other words, it is precisely the spacing between the altitudes which lies at the heart of the template; the angle of the first hypotenuse is merely a method for determining the proper spacing.

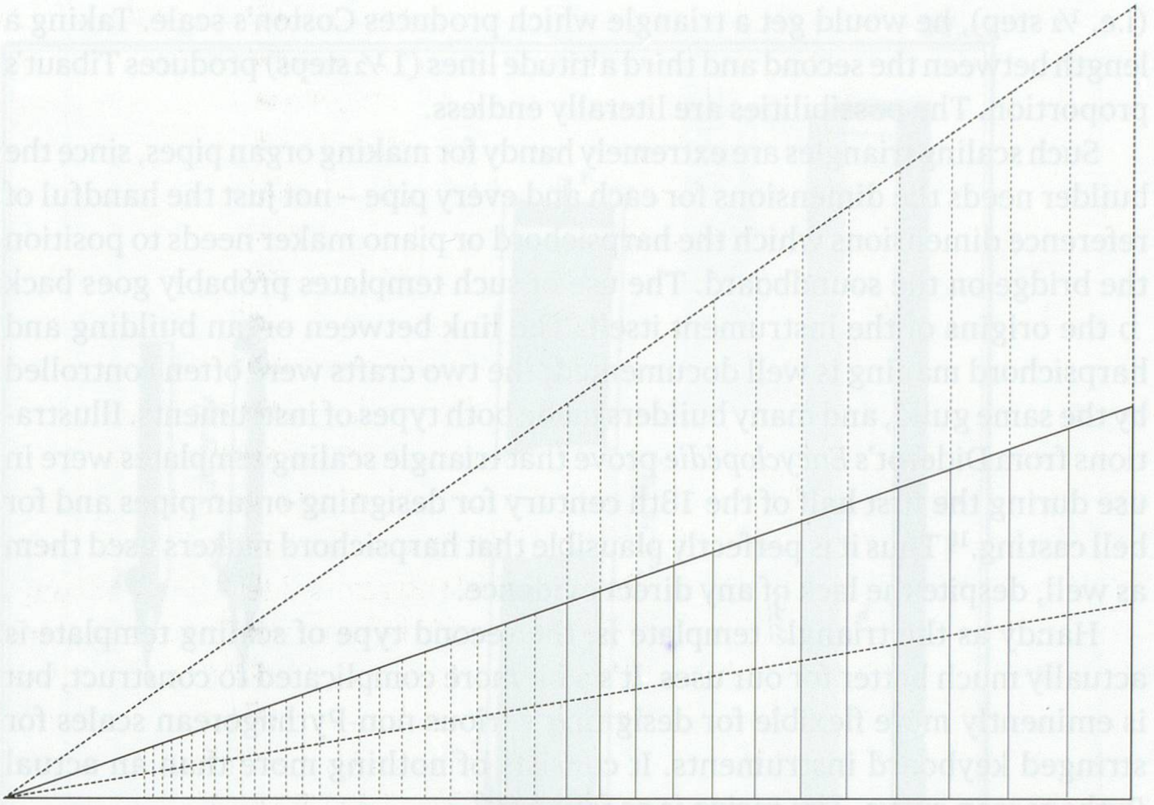

Figure 2 – Various methods for extending the range of the basic template.

ways by which a builder might decide upon alternative proportions are limited only by imagination. One possibility is to choose different integer values for the base-altitude ratio. For example, a triangle with the proportions of 28:9 would give us Tibaut's scale. Perhaps such simple integer pairs were part of the knowledge passed down from master to apprentice, much in the same way that the 3:4:5 right triangle recipe has been known in the building trades since the construction of The Pyramids. Another possibility is to use an existing Pythagorean template to determine irrational lengths for constructing a second non-Pythagorean template. In this case, the builder might begin with the idea that he wants a series of lengths which increase at a slightly less rapid pace. In other words, his octave ratio should be somewhat less than 2:1, somewhat closer to 1:1. This means the hypotenuse of the triangle should rise at a shallower angle, since the shallower the angle, the closer is the length of the hypotenuse to the length of the base. Therefore, the initial altitude should be slightly less than the altitude of the Pythagorean triangle. Keeping the same 20 unit length for the base, he could take a new "reduced" altitude from the existing template, specifying the amount of "scale alteration" by thinking in whole or half steps already defined according to Pythagorean proportions. For example, taking an altitude length exactly half way between the first and second altitude lines (i.e. ½ step), he would get a triangle which produces Coston's scale. Taking a length between the second and third altitude lines (1 ½ steps) produces Tibaut's proportion. The possibilities are literally endless.

Such scaling triangles are extremely handy for making organ pipes, since the builder needs the dimensions for each and every pipe – not just the handful of reference dimensions which the harpsichord or piano maker needs to position the bridge on the soundboard. The use of such templates probably goes back to the origins of the instrument itself. The link between organ building and harpsichord making is well documented; the two crafts were often controlled by the same guild, and many builders made both types of instruments. Illustrations from Diderot's *Encyclopédie* prove that triangle scaling templates were in use during the first half of the 18th century for designing organ pipes and for bell casting.[18] Thus it is perfectly plausible that harpsichord makers used them as well, despite the lack of any direct evidence.

Handy as the triangle template is, the second type of scaling template is actually much better for our uses. It's a bit more complicated to construct, but is eminently more flexible for designing various non-Pythagorean scales for stringed keyboard instruments. It consists of nothing more than an actual Pythagorean curve. The recipe is as follows:[19]

(1) Prepare a board a little larger than about 4 feet by 2 feet (grain running in the long dimension) by planning it smooth, shooting one long edge absolutely true, and then cutting one end perfectly square to this edge.

(2) Mark a series of short (1inch or so) evenly-spaced lines along the square end. The spacing between these lines is not critical as long as it is consistent; however, a spacing of ½ inch is both convenient and similar to the stringband spacing the template will be used to design. On a board which is 2 feet+ wide, you'll be able to rule out about 4 octaves worth (i.e. 24 lines per foot).

18 The authors of the text (pp. 943–4) accompanying the organ scaling plate, "MM. Thomas & Grossier", primarily offer an irregular division of the base line of the triangle using simple monochord (harmonic) proportions – neither a regular Just scale (12th root of 2) nor a constant-proportion non-Pythagorean geometric division, both of which I describe. They also offer two alternative tables of numerical values for dividing the base line, one for a *système tempéré*, the other ostensibly regular Pythagorean (equal) scaling (*la partition de l'octave en douze demi-sons éqaux*). Unfortunately, there are some fatal problems with the information: (1) the musical staff is printed upside-down, causing the relationship between numerical values and notes on the staff to be reversed; (2) in both the original and inverted form, the tables give neither a believable "temperament", i.e. one which even remotely resembles a meantone, a meantone variant (Rameau, Couperin, Schlick, or Werckmeister "modified" meantones), or any of the known well-temperaments, nor an "equal" division (i.e., equal temperament). Thus, like so much in the *Encyclopédie*, it appears that while the basic idea is correct, certain essential details are totally wrong.

19 I use the old system of feet and inches (12 inches = 1 foot) here – not the modern metric system – precisely because this is how old builders worked, regardless of what names they used for "feet" and "inches".

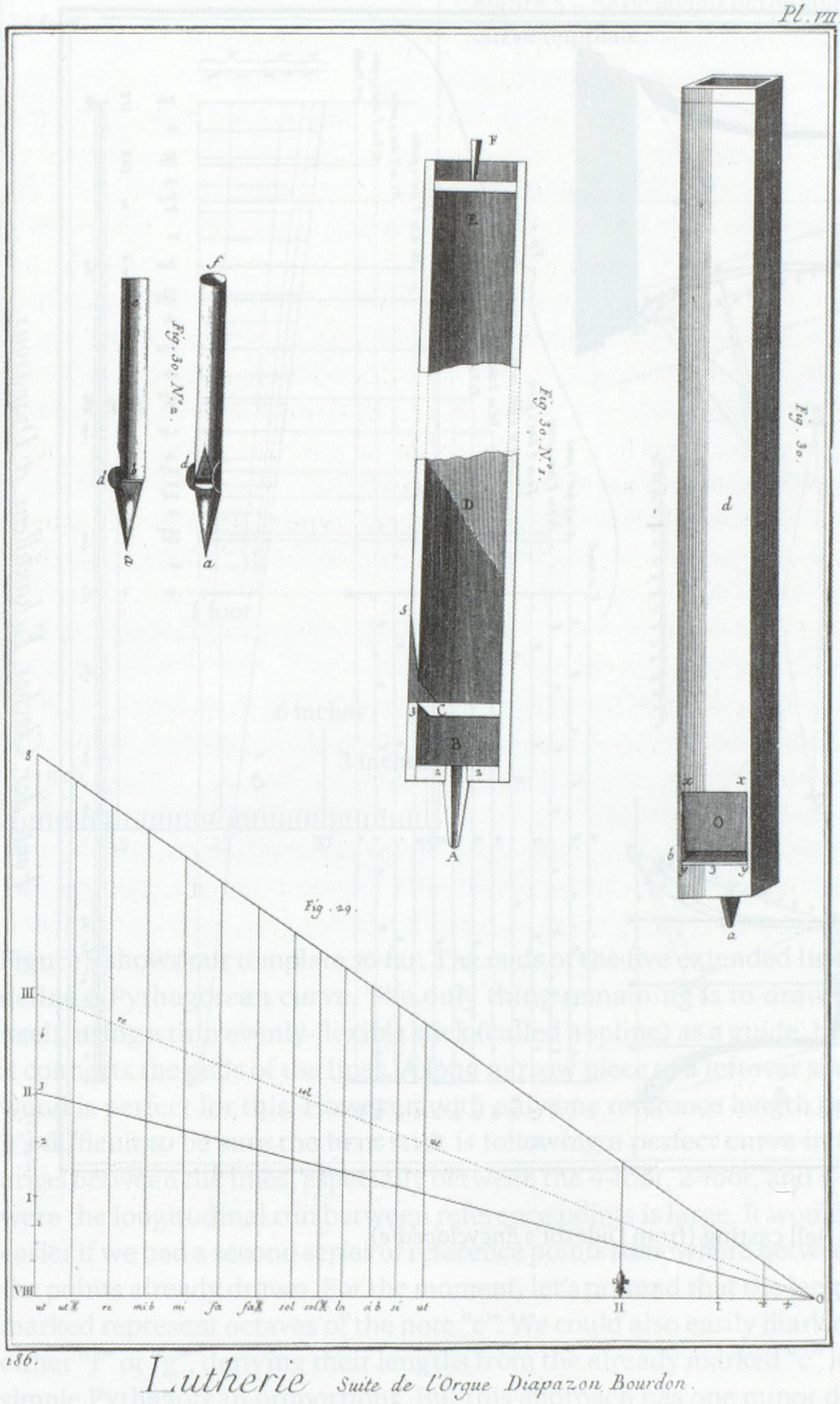

Figure 3 – Designing organ pipes (from Diderot's *Encyclopédie*).

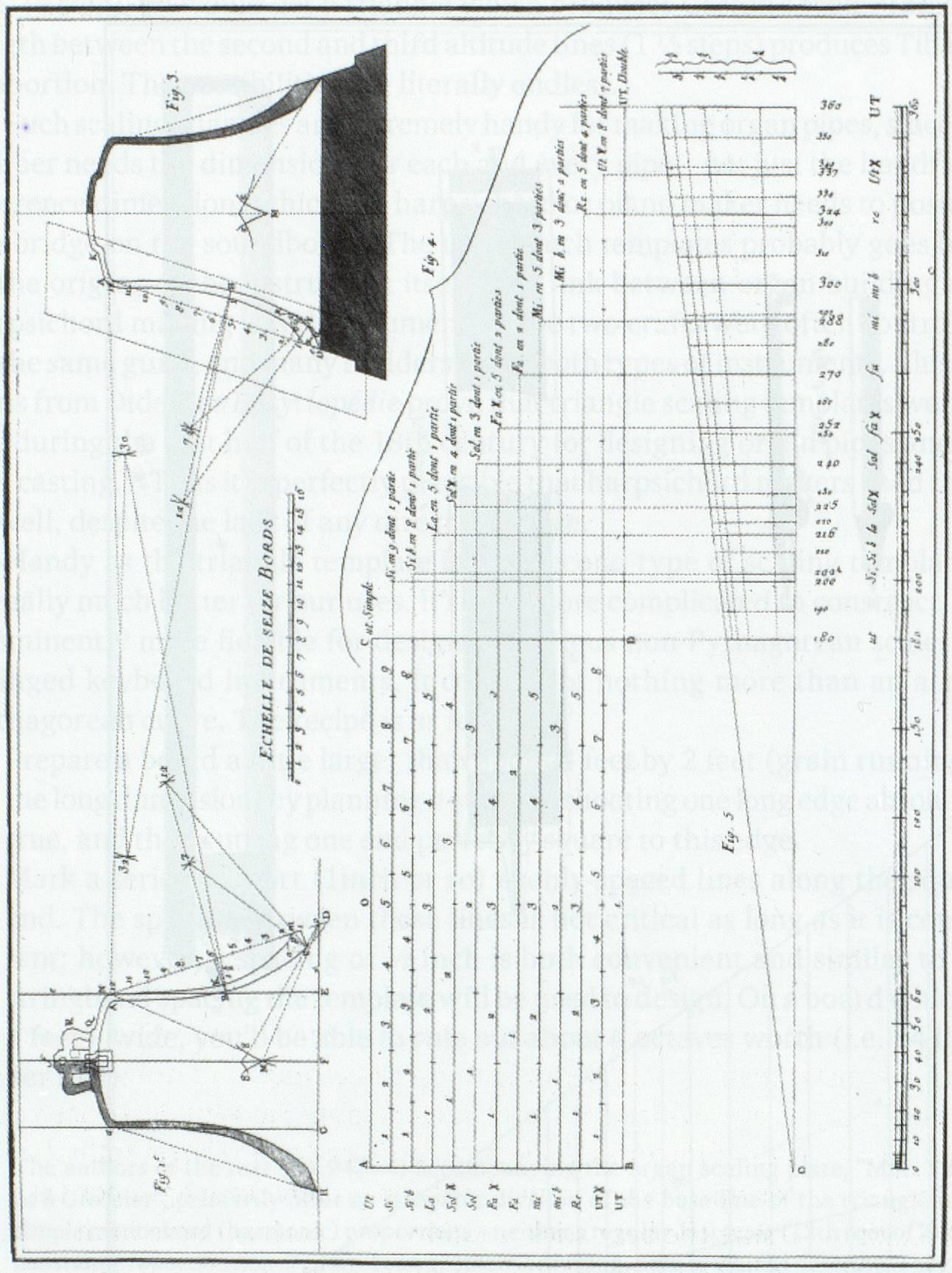

Figure 4 – Bell casting (from Diderot's *Encyclopédie*).

(3) Extend the first line to 4 feet from the end, the 13th line to 2 feet, the 25th line to 1 foot, the 37th to 6 inches (½ foot), and finally the 49th to 3 inches (¼ foot), being sure to keep all lines absolutely parallel to the true edge (and to one another).

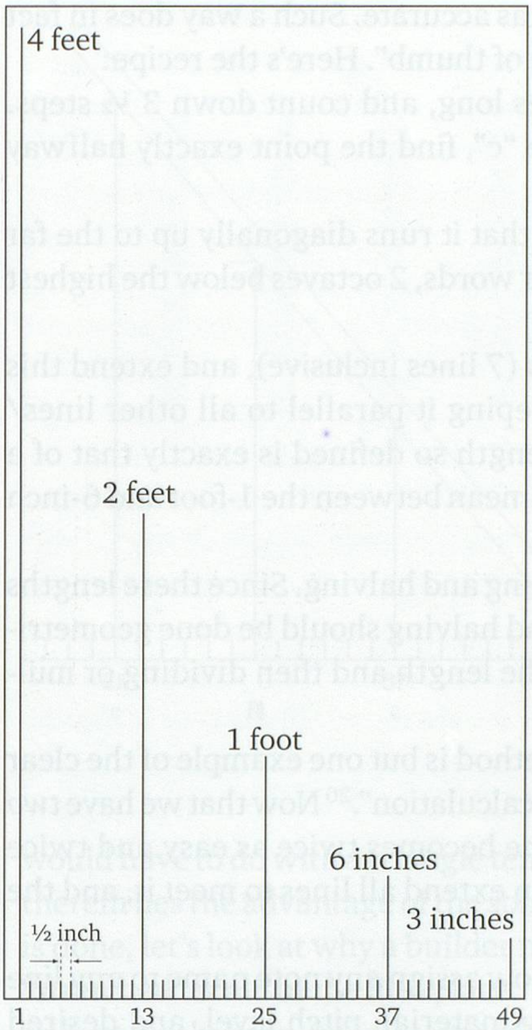

Figure 5 – Basic layout of the spline curve template.

Figure 5 shows our template so far. The ends of the five extended lines already define a Pythagorean curve. The only thing remaining is to draw the curve itself, using a thin evenly-flexible stick (called a spline) as a guide, bent so that it connects the ends of the lines. A long narrow piece of a leftover soundboard wood is perfect for this. However, with only one reference length per octave, it's difficult to be sure the bent stick is following a perfect curve in the broad areas between the lines, especially between the 4-foot, 2-foot, and 1-foot lines were the longitudinal run between reference points is large. It would be much easier if we had a second series of reference points somewhere between each of the points already drawn. For the moment, let's pretend that the lengths we've marked represent octaves of the note "c". We could also easily mark octaves of either "f" or "g", deriving their lengths from the already marked "c" lengths by simple Pythagorean proportions. But this approach has one minor drawback; either "f" or "g" will be closer to one or the other of the previously marked "c" lines. It would be nicer to have additional marks which fall exactly in the middle between the "c" octaves, i.e. at "f♯" (Ruckers again). We might mark the "f" and "g" and use them both, but if there were some fast and easy way to derive a perfect "f#", it would save time and be just as accurate. Such a way does in fact exist, again using a clever geometric "rule of thumb". Here's the recipe:

(1) Start at the 49th line, the one 3 inches long, and count down 3 ½ steps. In other words, if we call the 49th line "c", find the point exactly halfway between "g#" and "a".

(2) From this point, lay a straight edge so that it runs diagonally up to the far end of the 1-foot line (line 25) – in other words, 2 octaves below the highest (3-inch) "c".

(3) From the 1-foot line, count up 6 spaces (7 lines inclusive), and extend this line until it hits the straight edge (keeping it parallel to all other lines/ perpendicular to the base line). The length so defined is exactly that of a Pythagorean "f#", that is, the geometric mean between the 1-foot and 6-inch lengths.

(4) Mark all the other "f#" lengths by doubling and halving. Since these lengths will be irrational numbers, doubling and halving should be done geometrically, *not* by using a ruler to measure the length and then dividing or multiplying the numerical value.

Fast, easy, and incredibly accurate, this method is but one example of the clear superiority of geometrical over numerical "calculation".[20] Now that we have *two* marks per octave, the bending of the spline becomes twice as easy and twice as accurate. After tracing the curve, we can extend all lines to meet it, and the template is complete.

As with the triangle template, we can now assign any note name to any line on the template – depending on stringing material, pitch level, and desired wire stress level – and then derive the lengths of other reference notes merely by counting up or down. Better yet, we can make a 4 octave "note ruler", a long stick marked out with the same spacing as the lines on the template, labeled with the note names of the keyboard. This ruler can be slid left or right on the template, the starting note name aligned with the desired starting length, and the lengths of all other notes rapidly located for transferring onto a scaling stick. As with the triangle template, the lengths we end up with will usually be irrational numbers, not expressible as integer or simple fraction values of any local measuring system.

Now this is all very fine and good for calculating Pythagorean scales, but what about the *non*-Pythagorean proportions seen in the instruments above? Do we have to draw a new spline curve for each octave proportion, like we

20 I have made no attempt to investigate whether or not this method can be found in old treatises, nor do I consider it very important. The ratio of 41:29 (or as actually applied here, 20 ½:14 ½) is a very good approximation of the square root of 2, and its derivation is well within the capabilities of ancient monochord theorists. The geometrical method itself is quite easy to discover by simple trial and error, exactly as I did after only about 10 minutes of playing around with the scaling template.

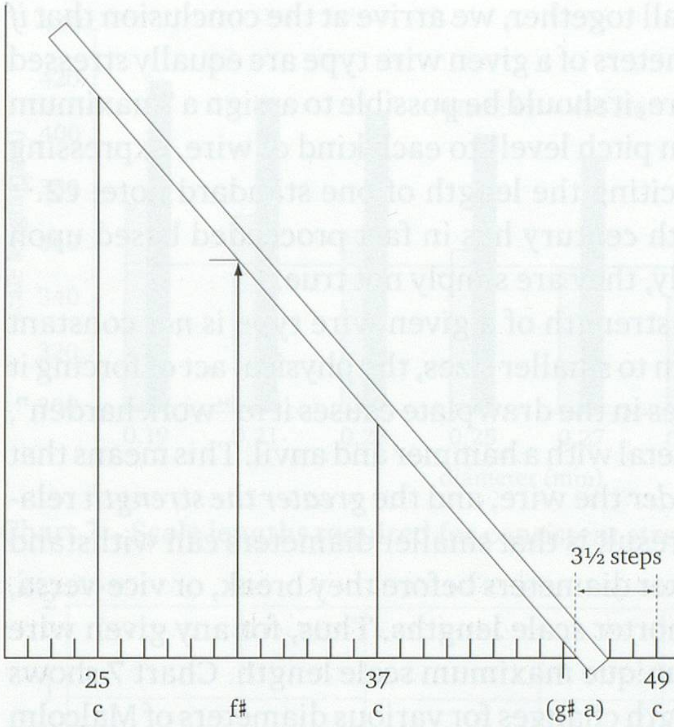

Figure 6 – Geometric method for finding the length of f♯.

would have to do with a triangle template? Surprisingly, the answer is "no", and therein lies the advantage of the spline curve template. Before I discuss how this is done, let's look at why a builder might want a non-Pythagorean "tapered" or "stretched" scale in the first place. The reasoning behind stretched scaling has only recently been (re)discovered, although there is ample evidence that old builders knew about it.[21]

Modern organologists have mostly assumed that wire strength is "material specific", that is, the same for all diameters (relative to the cross sectional area) of any given type – iron, brass, etc. Were this true, the actual diameters which were going to be used on an instrument would not be important consideration during the initial design phases when determining the scale length. A heavier stringing would require more tension than lighter strings, requiring a heavier case structure, but in terms of wire stress and pitch handling capability, the increased tension would be perfectly balanced by the increased capability of thicker strings to withstand a higher load. In other words, organologists usually assumed that all diameters of a given wire type would break at the same pitch when mounted on a given length.

The different lengths used for the different notes on a Pythagorean scale would also not change the stress level, because the *increase* in length (going down) is always perfectly balanced by the *decrease* in pitch, keeping tension (for

21 See Goodway/Odell, pp. 61–63 for quotes from Adlung and Corrette. Julius Blüthner, in his *Lehrbuch des Pianofortebaues* (Leipzig, 1872) also recommends stretching the scale in the treble "as much as the quality of the wire will allow".

each diameter) equal. Putting it all together, we arrive at the conclusion that *if* the scale is Pythagorean, *all* diameters of a given wire type are equally stressed *everywhere* on the scale. Therefore, it should be possible to assign a "maximum breaking scale length" (at a given pitch level) to each kind of wire, expressing these absolute scale lengths by citing the length of one standard note: c2.[22] Much organology of the late 20th century has in fact proceeded based upon these assumptions. Unfortunately, they are simply not true.

The truth is that the relative strength of a given wire type is *not* constant for all diameters. As wire is drawn to smaller sizes, the physical act of forcing it through successively smaller holes in the drawplate causes it to "work harden", exactly as when we "cold form" metal with a hammer and anvil. This means that the *smaller* the diameter, the *harder* the wire, and the *greater the strength* relative to cross-sectional area. The result is that smaller diameters can withstand a longer absolute scale than thicker diameters before they break, or vice-versa, larger diameters will break at shorter scale lengths. Thus, for any given wire type, *each diameter* has its own unique maximum scale length. Chart 7 shows how much the breaking scale length changes for various diameters of Malcolm Rose "English Iron Type A" wire. Much as we would like, we simply cannot talk about the stress level of a certain wire type at a given pitch level and scale length *unless* we also specify the diameters being used. This inescapable fact means that many of the conclusions of 20th century organology about pitch level and scale length will have to be reexamined.

One such conclusion is the idea that ancient builders chose scale lengths which produced stress levels that were very close to breaking. This idea can only be true under a limited set of circumstances, conditions which are in fact *not true* for many instruments. The problem arises from the ubiquitous use of progressively smaller diameters in the treble. If the scale is Pythagorean, the inescapable result is that wire stress drops significantly in the treble. These two graphs illustrate the effect of scale shape on stress level, using the degree of "tensile strength pick-up" exhibited by Rose "A" iron.[23]

22 The system is further "enhanced" by using "c2 equivalents" to express the lengths of notes other than c2. This number is the length of the note c2 in whatever Pythagorean scale the note in question would be found.

23 Little is known about the "tensile strength pick-up" rate of 17th and 18th century wire, though there is no reason to suspect it would have been radically different from modern ferrous wire. To my knowledge, only one test has been done on several diameters of wire known to be of consistent origin and assumed to be pre-1800, by Michael Latcham and Alphons Huber using strings from a Hofmann piano; see Latcham, *The Stringing, Scaling and Pitch of Hammerflügel built in the Southern German and Viennese Traditions 1780–1820*, München-Salzburg (Katzbichler) 2000, p. 86 and table 74. This test indicates a slightly higher pick-up rate than for modern Rose wire, which would require an even greater degree of scale stretching to maintain consistent stress among all diameters.

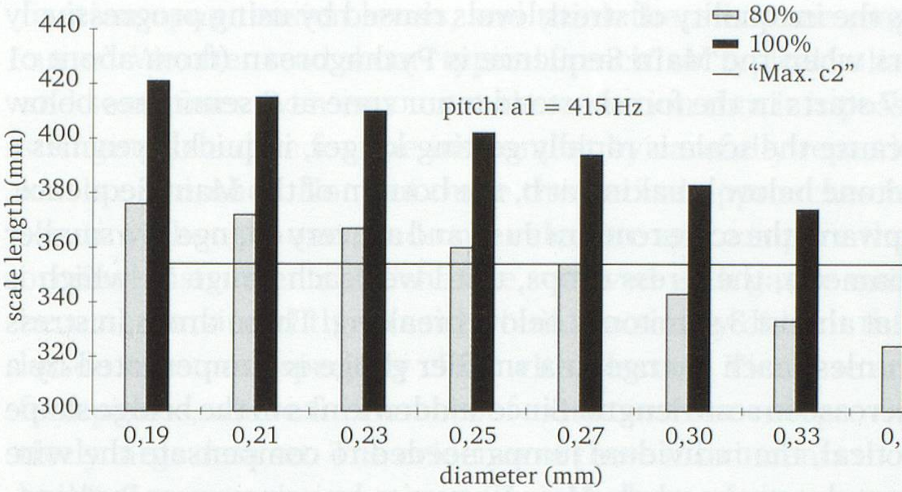

Chart 7 – Scale lengths required for consistent stress levels for Rose "A" iron wire.

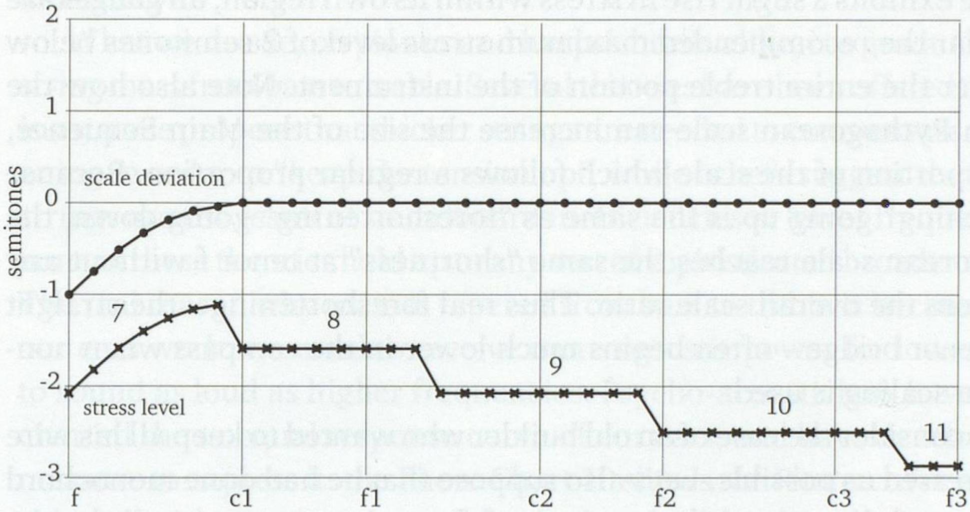

Chart 8 – Inequality of stress levels caused by using progressively smaller string diameters (gauge 7 to 11) in a Pythagorean Main Sequence.

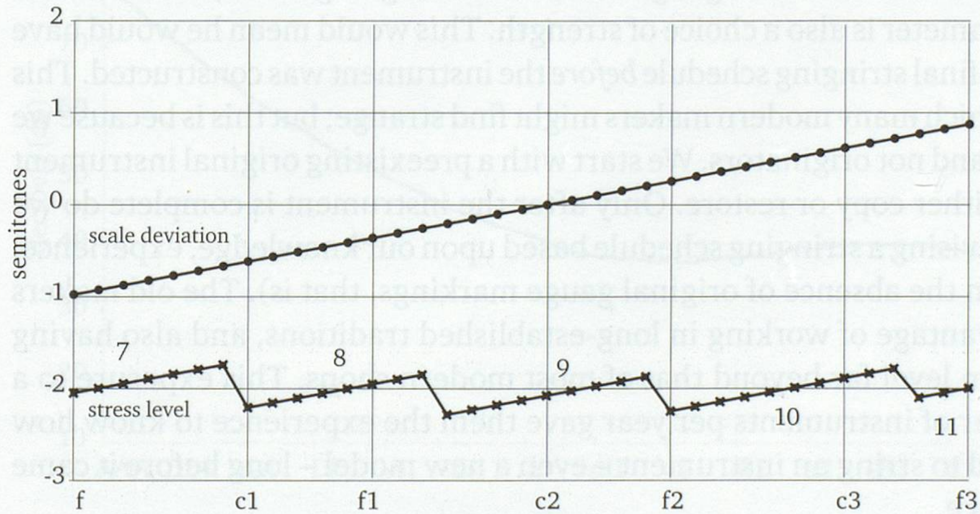

Chart 9 – Inequality of stress levels caused by using progressively smaller string diameters (gauge 7 to 11) in a non-Pythagorean Main Sequence (octave ratio 1:1.93).

Chart 8 shows the inequality of stress levels caused by using progressively smaller diameters when the Main Sequence is Pythagorean (from about c1 upward). Gauge 7 starts in the foreshortend tenor zone at 2 semitones below breaking, but because the scale is rapidly getting longer, it quickly reaches a dangerous 1 semitone below breaking at b, the bottom of the Main Sequence. From this point upward, the scale remains Just, and at every change to a smaller (and stronger) diameter, the stress drops, until we reach gauge 11 which is "under-stressed" at almost 3 semitones below breaking. These drops in stress are unavoidable unless each change to a smaller gauge is compensated by a corresponding increase in scale length. Since sudden kinks in the bridge shape are not very practical, the individual jumps needed to compensate the wire changes are averaged over the whole Main Sequence by using a non-Pythagorean octave ratio (Chart 9), in this instance 1:1.93. Note that while each individual gauge exhibits a slight rise in stress within its own region, all gauges are now kept near the recommended maximum stress level of 2 semitones below breaking over the entire treble portion of the instrument. Note also how the use of a non-Pythagorean scale can increase the size of the Main Sequence, that is, that portion of the scale which follows a regular proportion. Because scale "stretching" going up is the same as "foreshortening" going down, the non-Pythagorean scale reaches the same "shortness" at tenor f without any deviation from the overall scale ratio. Thus real foreshortening – the straight run of the tenor bridge – often begins much lower in the compass when non-Pythagorean scaling is used.

So – let's consider the case of an old builder who wanted to keep all his wire as evenly stressed as possible. Let's also suppose that he had done monochord experiments and discovered the strength pick-up phenomenon, and that he therefore knew that the only way to keep stress constant is by using a stretched scale. How would he go about *designing* such a scale? Naturally, the first thing he would need to know is which gauges of wire he was going to use, because each choice of diameter is also a choice of strength. This would mean he would have to know the final stringing schedule *before* the instrument was constructed. This is an idea which many modern makers might find strange; but this is because we are copiers and not originators. We start with a preexisting original instrument which we either copy or restore. Only after the instrument is complete do we set about devising a stringing schedule based upon our knowledge, experience, and taste (in the absence of original gauge markings, that is). The old makers had the advantage of working in long-established traditions, and also having a production level far beyond that of most modern shops. This exposure to a high number of instruments per year gave them the experience to know how they wanted to string an instrument – even a new model – long before it came into existence.

How might a maker establish a stringing schedule for an as yet non-existent instrument? Some organologists adopt the late-19th century piano builder's fixation upon tension levels, and think there is some magic "ideal tension curve". We often read about "typical French" or "typical Italian" tension curves, and sometimes even encounter the assertion that it is possible to devise the stringing schedule for one instrument by carefully reproducing the tension levels of another. None of these theoretical approaches hold up to careful scrutiny, though; one need only calculate and compare the tension levels of original instruments which have survived with gauge markings in order to become convinced that ancient builders simply did not think this way. This lesson becomes especially clear with Classical Viennese and South-German fortepianos, where the number of extant instruments provides us with a relatively large data set. The evidence of these instruments shows that while old builders were experimenting with scale *shape*, there appears to be no attempt whatsoever to keep tension and/or stress levels in agreement with any sort of ideal curve.

What we do see, on pianos and harpsichords alike, is a gradual reduction of string mass from bass to treble. Several theories have been offered up as why this is so; some propose that thicker strings are needed to compensate for foreshortening in order to "keep the tension up", while others suggest that thicker bass strings are the result of soundboard inefficiencies at lower frequencies. Again, none of these theories hold up to any sort of sophisticated broader examination. The real reason why mass is tapered from bass to treble is because that is how our ears work: we need to receive more acoustic power for lower frequencies to sound as loud as higher frequencies. Psycho-acousticians have studied and charted the unequal response curve of our ears, and have developed a unit of measure to express it: the Phon. Chart 10 shows the Phons curve at a moderate volume level over the range of the modern keyboard:

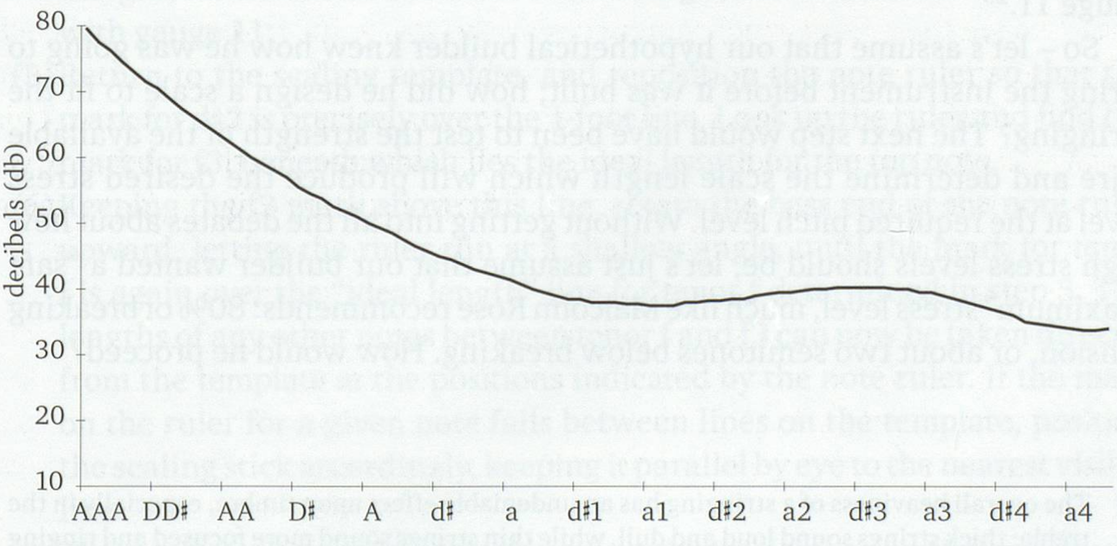

Chart 10 – Sound pressure level for equal perceived volume level (Phons).

The similarity of this power curve to the ubiquitous distribution of stringing mass should be immediately obvious. Basses are always strung with strings significantly thicker than in the treble, reducing rapidly at first before leveling off somewhat for the upper half of the instrument. Other factors unique to the basic design of each instrument, such as soundboard resonances and efficiencies, case resonances, and overall timbre preference[24] may well contribute to each builder's final stringing solution. However, if a maker does not generally taper mass so as to follow the Phons curve, the various registers of an instrument will sound out of balance with one another.

As we might expect, old builders devised innumerable ways of distributing mass so as to approximately balance the response curve of our ears. Some simply used equally-sized groups of gauges, smaller groups in the bass and larger groups in the treble. Such a stringing might begin with several groups of two notes per gauge in the deep bass, followed by several slightly larger groups of three notes/gauge in the low tenor, and then suddenly jumping to much larger blocks of eight (or even nine or ten) notes/gauge from about tenor f up (the point at which the Phons curve becomes relatively level). The success of such a semi-linear approach was enhanced by the logarithmic reduction of diameter which characterized most ancient gauge systems; the *linear* distribution of such gauges produced a *logarithmic* reduction in mass. Other builders "tapered the taper", gradually increasing the size of each gauge block from bass to treble. One particularly clever example of how this could be done was devised by J.H. Silbermann; he simply let the gauge number dictate the number of notes strung with that gauge. He started by tapering the bass rapidly, using one note each of gauge 3/0, 2/0, and 0, and then continued with one note of gauge 1, two notes of gauge 2, three notes of gauge 3, etc, all the way up to ten notes of gauge 10, until the system finally "fell off the edge of the world" after only four notes of gauge 11.[25]

So – let's assume that our hypothetical builder knew how he was going to string the instrument before it was built; how did he design a scale to fit the stringing? The next step would have been to test the strength of the available wire and determine the scale length which will produce the desired stress level at the required pitch level. Without getting into all the debates about how high stress levels should be, let's just assume that our builder wanted a "safe maximum" stress level, much like Malcolm Rose recommends: 80% of breaking tension, or about two semitones below breaking. How would he proceed?

24 The overall heaviness of a stringing has an undeniable effect upon timbre, especially in the treble; thick strings sound loud and dull, while thin strings sound more focused and ringing though less loud. These general differences are caused primarily by stiffness as it is affected by the diameter/length ratio.

25 I am indebted to John Phillips for pointing this out to me.

This is where we see how a Pythagorean spline curve template can be used to design a non-Pythagorean scale. Here's the process a builder might follow:

(1) Position the bridges of a monochord at a length which matches a length on the scaling template, somewhere near the middle. Which length one chooses is not important, though one too long would waste wire and one too short would be difficult for testing large diameters. Let's take a length of 1 foot. Exactly how long that particular "foot" may have been is completely irrelevant, as long as the template was constructed using the same units of feet and inches.

(2) Decide which note will define the bottom of the main sequence – say tenor f – and take the gauge of wire to be used there – say gauge 7. Mount a string of this size on the monochord and tune it up until it breaks, continuously comparing the pitch against an external pitch source – a pitch pipe, an organ – whatever, as long as it gives the pitch level desired for the instrument. Suppose the string breaks at about d♯2. Count down two semitones for the desired safety margin, and we find that 1 foot is the ideal length for the note c♯2 when strung with gauge 7.

(3) Position the note ruler on the template so that the mark for c♯2 is precisely above the 1-foot line, and then look down the ruler and find tenor f. Beneath this mark lies the line with the ideal length for tenor f when strung with gauge 7 – that is, the length that will give the desired two semitone safety margin. Make some sort of mark on this line so that it can easily be found again later.

(4) Using the same 1-foot length on the monochord, repeat the strength test with the thinnest gauge to be used, gauge 11. This string, being relatively stronger, will break at a slightly higher pitch – let's say it breaks at f2, two semitones higher than the gauge 7 string. Counting down again for safety margin, we learn that 1 foot is the ideal length for the note d♯2 when strung with gauge 11.

(5) Return to the scaling template, and reposition the note ruler so that the mark for d♯2 is precisely over the 1-foot line. Look up the ruler and find the mark for f3, beneath which lies the ideal length for the top note.

(6) Keeping the f3 mark above this line, rotate the bass end of the note ruler upward, letting the ruler run at a shallow angle, until the mark for tenor f is again over the "ideal length" line for tenor f determined in step 3. The lengths of any other notes between tenor f and f3 can now be taken directly from the template at the positions indicated by the note ruler. If the mark on the ruler for a given note falls between lines on the template, position the scaling stick accordingly, keeping it parallel by eye to the nearest visible line.

The whole process takes less than ten minutes (assuming the template is already made). Even we began with a rational length of 1 foot, *all* of the resultant string lengths are irrational numbers, and will confound any attempts to

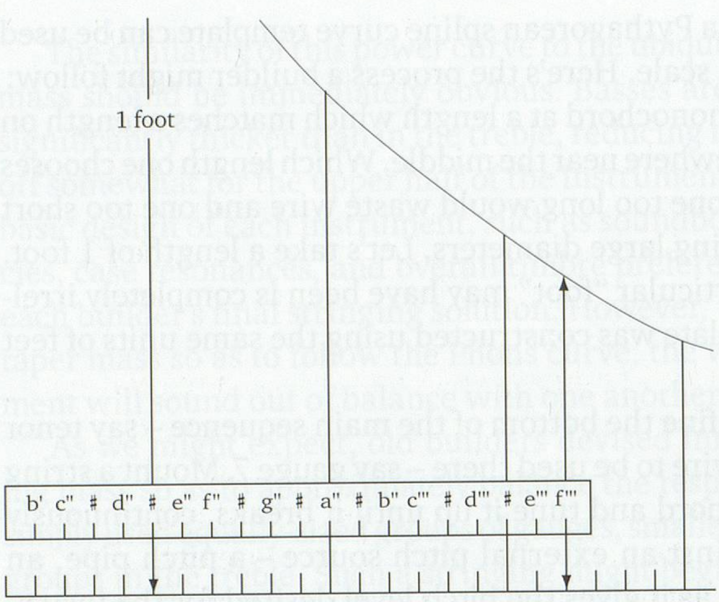

Figure 7 – Using the note ruler to find the length of f''' on the template.

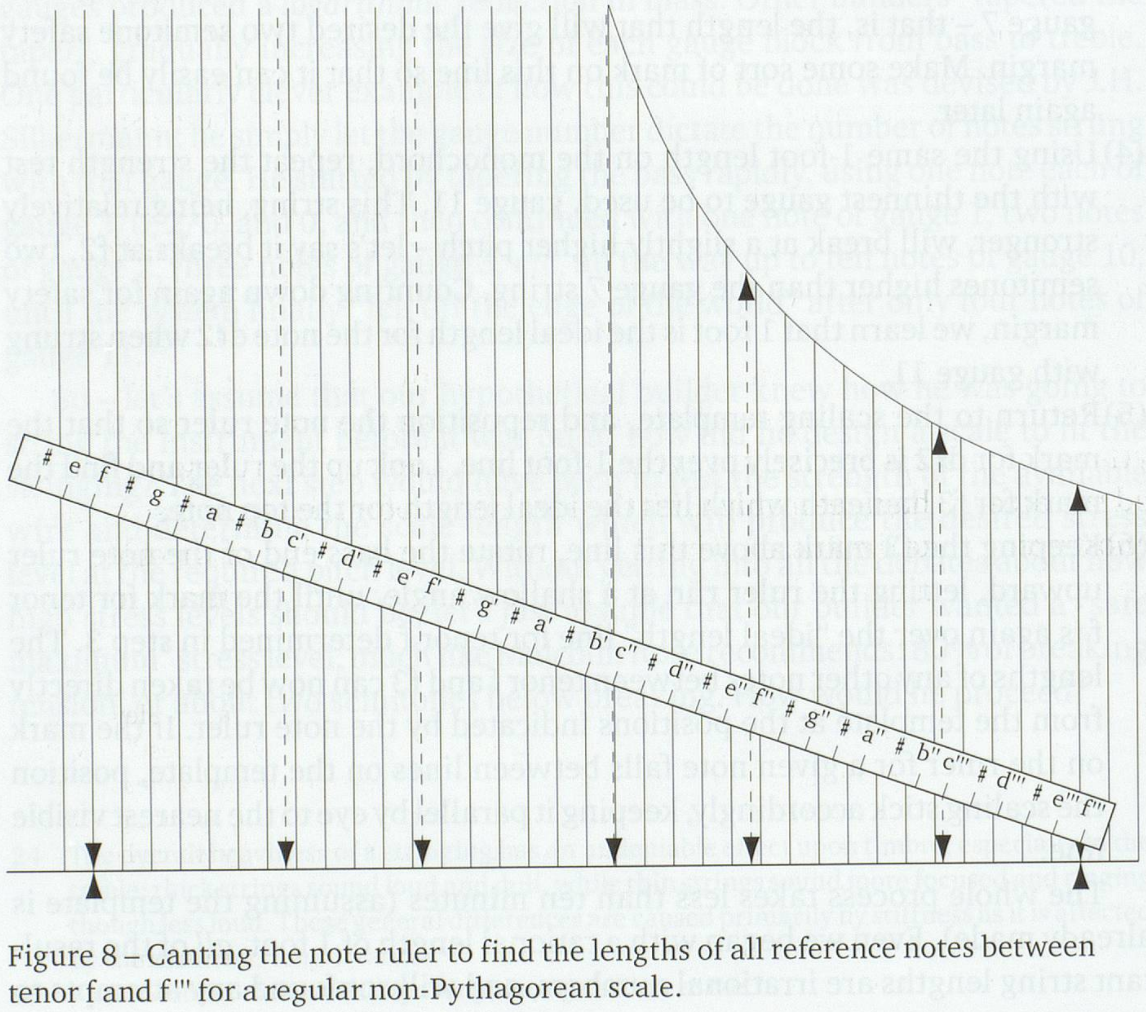

Figure 8 – Canting the note ruler to find the lengths of all reference notes between tenor f and f''' for a regular non-Pythagorean scale.

subjugate them to the Tyranny of the Whole Zoll. Using the simplest of tools and no mathematics whatsoever, our builder has designed a regular logarithmic non-Pythagorean scale perfectly suited to the stringing and pitch circumstances of the particular instrument. Should he later decide to alter his distribution of mass, to increase or decrease overall stringing weight, to make an instrument for another pitch level, or use wire from a new unfamiliar source, he simply repeats the process again, perhaps coming up with a slightly different tapering scheme.[26]

We now have a reason for *why* a builder would want to use a regular non-Pythagorean scale as well as a holistic methodology for *how* he could design and apply such a scale. In this light, we might well ask ourselves, "Why aren't *all* scales non-Pythagorean?" In the case of instruments strung wholly in brass, this may be due to the fact that the amount of strength increase exhibited by brass wire is much lower than that for iron, and the amount of scale stretching made possible by successively thinner treble strings is almost nil. This supports the usual assumption that scales which are completely Pythagorean (excepting a few notes at the bottom) were intended to be strung entirely in brass. Nonetheless, some instruments assumed to have been strung in brass do indeed have stretched scales. The Beurmann Sodi[27] has a large Main Sequence from bass C to f2 which follows the non-Pythagorean proportion of 1:1.91. Above f2, the scale is stretched even more, being 2 semitones longer at f3. Perhaps this instrument was not brass strung at all, but rather iron strung at a high pitch.

In addition to the instruments mentioned here, there are other harpsichords from northern building traditions which also have Main Sequence scales which deviate significantly from the assumed norm of Pythagorean proportions. Two late French instruments by Taskin, for example, actually have a significantly stretched scale, though not of the continuous type seen with Tibaut and Gräbner. The Russell Collection instrument is Pythagorean for only 1 ½ octaves, from middle c to f2. By c3, the scale has been stretched by 1 semitone, and at f3, by two semitones. The Portuguese Taskin[28] has a very similar scale, except that the Pythagorean area is even smaller yet; the Main Sequence doesn't begin until f1, and remains Pythagorean only until f2, from which point the scale is stretched almost exactly like the Russell. Thus Taskin was in fact stretching his treble scale by an amount similar to Tibaut and Gräbner, except through a different means.

26 My method is essentially that described by the Viennese piano maker Jakob Bleyer in a newspaper article of 1811. He began by using a monochord to determine the "best" lengths for tenor f and f4, between which he derived the lengths of the intervening notes so that they described a "geometric" (i.e. logarithmic) progression. See Latcham, *op. cit.*, p. 60 for Bleyer's text. The only thing I have added to Bleyer is a practical method by which he could have easily determined the intervening notes.

27 Now in the Museum für Kunst und Gewerbe, Hamburg, Inv. No. 2000.521.

28 See Bernard Brauchli, *The 1782 Taskin Harpsichord, Colares, Portugal*, The Galpin Society Journal LIII (April 2000), pp. 25–50.

His practical methodology could also have involved the use of a scaling template of either kind, deriving the stretched lengths for c3 and f3 by adding one and two *semitones* respectively, i.e. by taking the theoretically-correct lengths for b2 and d#3 (relative to his c1 or f1 length) directly from the template.

Despite all this, many instruments do exhibit pure Pythagorean scaling over the entire Main Sequence. The unavoidable conclusion is that some builders simply did not care about keeping wire consistently close to breaking, and that they were quite happy to allow their treble strings to become gradually "under-stressed". This flies in the face of much of modern organology, but the truth of the matter is undeniable. However, when we reexamine the scaling of extant instruments using analysis methods which more clearly illustrate aberrations from strict Pythagorean curves, we may well find that the number of instruments which fit this standard assumption is smaller than we now think.

Finally, I want to stress one point – in fact, if you remember but one thing from this whole workshop, I would hope that it were this. Look again at the two graphs (Charts 8 & 9) showing the effect of scale shape upon stress levels. In both cases the hypothetical scale length is c2 = 360 mm. Note that the stretched instrument could be tuned one semitone higher than the Pythagorean instrument before it would reach the same maximum stress level. Looking at it the other way around, the Pythagorean instrument would have to be pitched one semitone lower in order to have the same maximum stress level. This demonstrates a very important phenomenon: *if the scale is Pythagorean, the most dangerously stressed note will always be near the bottom of the Main Sequence*, usually middle c or tenor f – *not* c2! The length of c2 will not reflect the stress level found in the tenor; unless the scale is stretched, c2 will always be at a *lower* stress level, often significantly lower. The length of c2 is therefore completely untrustworthy as an overall indicator or shorthand expression for matters of pitch/scale/wire-type, despite the fact of its ubiquitous use therefore. Much as we would like it to be so, the questions of pitch level, stress level, and scale length are not as simplistic as traditional organology would have us believe, and the probability is high that a number of past conclusions based upon the use of c2 lengths will eventually be proven to be untrue. Perhaps the only thing we can say with any certainty is that a complete reexamination of the topics of stringing, scaling, and pitch of historical keyboard instruments will undoubtedly hold some surprises for us.